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### Lessons to develop some problem solving strategies for high school students of varying abilities

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*The University of Montana*

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LESSONS TO DEVELOP SOME PROBLEM SOLVING STRATEGIES  
FOR HIGH SCHOOL STUDENTS OF VARYING ABILITIES

By

Diane E. Burrell

B.A., University of Montana, 1970

Presented in partial fulfillment of the  
requirements for the degree of  
Master of Arts for Teachers of Mathematics

UNIVERSITY OF MONTANA

1982

Approved by:

Shlomo Libeskind  
Chairman, Board of Examiners

Robert Murray  
Dean, Graduate School

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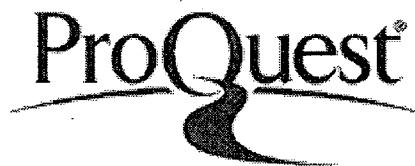


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Burrell, Diane E., MAT, Summer 1982      Mathematics

Lessons to Develop Some Problem Solving Strategies for High School Students of Varying Abilities (102 pp.)

Director: Shlomo Libeskind *SL*

This thesis presents mathematics problems to be used in teaching problem solving skills to high school students. Four of the chapters introduce specific problem solving techniques, examples of their use, suggestions for classroom presentation, and accompanying student exercises. The following six chapters present problems for the students to solve using these techniques. The heuristics of the solution and alternate methods of solution are discussed. The problems can be approached from many levels and can lead to sustained investigations in algebra, geometry, and number theory.

## INTRODUCTION

There is a strong movement among mathematics educators to emphasize problem solving in the mathematics curriculum. One of the recommendations of the National Council of Teachers of Mathematics is that "Problem solving must be the focus of school mathematics in the 1980s."<sup>1</sup> In a national survey of mathematics educators, problem solving was consistently ranked high in priority for increased emphasis in the 1980s. It received strong endorsement as the first priority for the development of new materials.<sup>2</sup>

With all this emphasis on the importance of problem solving, there is much confusion about how precisely to define problem solving, and how to go about teaching students a problem solving process. Most people will agree that a good problem is one which requires a student to "think" -- a very vague definition. More precise criteria for a problem can be derived from Bloom's Taxonomy of Educational Objectives.<sup>3</sup> Six major categories of cognitive thought processes are listed:

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<sup>1</sup> National Council of Teachers of Mathematics, An Agenda for Action: Recommendations for School Mathematics of the 1980s (Reston: NCTM Publications, 1980).

<sup>2</sup> National Council of Teachers of Mathematics, Priorities in School Mathematics (Reston: NCTM Publications, 1981), p. 29.

<sup>3</sup> Benjamin S. Bloom, ed., Taxonomy of Educational Objectives, vol. 1: Cognitive Domain (New York: David McKay Co., 1956), p. 201-207.

1. Knowledge
2. Comprehension
3. Application
4. Analysis
5. Synthesis
6. Evaluation

They are listed in a hierarchical order; the objectives in one class generally are built on the behaviors in the preceding classes.

One criterion that a good problem should meet is that it make use of at least one of the upper three cognitive levels, analysis, synthesis, or evaluation. Mathematics is perhaps the natural subject for teaching the process of analysis. To solve a math problem or to understand an algorithm, the student must analyze the elements of the problem and the relationships between them. Teaching a student how to approach systematically such an analysis is a primary problem solving goal.

Synthesis is the process of arranging elements and parts to produce a new pattern or structure. Again, mathematics is well suited to exercise this process through the use of mathematical generalizations. Generalizing requires a student to develop a new structure from the pieces of specific problems. Formulating hypotheses is also considered a type of synthesis, as is deriving a plan for attacking a particular problem.

Bloom defined two types of evaluation: evaluation in terms of internal evidence and evaluation in terms of external evidence. Any sort of proof requires the student to judge the validity of a logical



argument and therefore would constitute evaluation in terms of internal evidence. The comparison of theories or methods would require the use of evaluation in terms of external evidence.

The question of how to teach a student to solve problems is far more difficult than that of selecting good problems to solve. Alfred North Whitehead put forth some interesting thoughts on stages of learning. He maintained that there was a natural rhythm to the mastery of a subject, and that educators would be most successful if they adapted their teaching to this rhythm. He defined three stages which must be passed through before any material can be totally understood. These three stages he called romance, precision, and generalization. Romance is the initial fascination with a new problem. It is the process of discovery and the stage of questioning, experimenting, and exploring. Whitehead contends that it is the romance stage which provides the motivation for solving the problem. ". . . a stage of precision is barren without a previous stage of romance: unless there are facts which have already been vaguely apprehended in their broad generality, the analysis . . . is simply a series of meaningless statements about bare facts, produced artificially and without any further relevance."<sup>4</sup>

Precision is the stage most emphasized in schools. It is the process of an orderly acquisition and organization of facts and details. "The facts of romance have disclosed ideas with possibilities of wide

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<sup>4</sup> Alfred North Whitehead, Aims of Education and Other Essays (New York: Macmillan Company, 1929), p. 29.

significance, and in the stage of precise progress we acquire other facts in a precise order . . . ."<sup>5</sup>

Generalization is the stage which truly tests the mastery of the solution of a problem. To learn something permanent from a problem, one must be able to generalize the solution and apply it to other problems. Without this stage, students' problem solving abilities will not grow. They will remain stuck in the rut of seeing each problem as an isolated obstacle and not as part of the unified structure of mathematics.

The problems presented in this thesis are intended to be used with high school students. The solutions have been written with the assumption that they will be presented to students who have a knowledge of algebra and geometry. However, the problems have been specifically chosen to be adaptable to many levels and most of them could be modified for use at a lower level. All the problems chosen use at least one of the cognitive thought processes of analysis, synthesis, or evaluation, and Whitehead's three stages of learning have been utilized whenever possible.

There is a great deal of difference between solving problems and teaching problem solving skills. To have students practice solving problems, the teacher would simply present a problem and expect the students to sit down and solve it on their own. For most high school students, however, this would merely be an exercise in frustration.

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<sup>5</sup> Ibid., p. 30.

The purpose in teaching problem solving is to help students develop techniques and processes to use in attacking a problem. The techniques should be discussed at length, and students should have many opportunities to practice them. The first four chapters introduce the student to some basic problem solving techniques. These techniques are used in subsequent problems, and the students should be familiar with them so that it does not seem as though their use is artificial. Not all strategies that will be used are covered, only those which students probably have not discussed previously.

George Polya contends there is much to be learned from following the solution of a problem with genuine interest and insight. It certainly is not harmful to have the teacher demonstrate the solution to a problem as long as the emphasis is on the heuristics of the solution and not merely on the answer to the problem.

One strategy that works very well in teaching problem solving is to have the students work together in groups. In non-academic situations, most problem solving seems to be done by group interaction rather than individually. Access to several viewpoints can lessen student frustration and build confidence.

The successful teaching of problem solving is ultimately a function of the teacher. The teacher has to find the median between simply telling students the answer and letting them flounder in frustration. Any teacher must be flexible enough to be responsive to the needs of the class rather than to a preconceived lesson plan. The atmosphere should be open enough to allow students to volunteer their ideas with-

out fear of failure, for students can learn much from their mistakes if they are not embarrassed by them. Finally, the teacher must be enthusiastic about the problem and be willing to discuss the thought processes being applied. Teacher enthusiasm will override almost any flaw in the structure of the lesson, but teacher disinterest will endanger the spirit of romance necessary to any successful problem solving experience.

The author has indicated where she adapted methods, problems, and ideas from other sources. Although a particular problem may have come from another source, unless otherwise indicated, the development of the solution and the accompanying heuristics are due to the author.

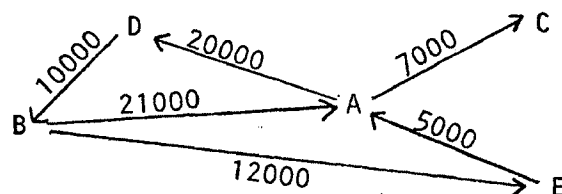
## PICTURE THIS

Drawing a diagram of a situation is one of the most basic problem solving techniques. In the elementary grades, pictures are heavily relied upon to explain and clarify problems. As students become more experienced, pictorial representations are used less. In high school math courses, diagrams are still used in geometric contexts, but their use is sometimes ignored in arithmetic or algebraic settings. This chapter illustrates the usefulness of a diagram in problems where the students must deal with a jumble of unorganized data.

The following problem is a good example:

Banks that are members of the same system often shift funds among themselves as needed. Each bank, however, is a separate entity and is responsible for its own funds. Call five banks in a system A, B, C, D, and E. On Monday, bank A loaned \$21000 to bank B. On Tuesday bank A experienced heavy demand and had to borrow \$7000 from bank C and \$20000 from bank D. Wednesday bank B borrowed \$12000 from bank E and bank D in turn borrowed \$10000 from bank B. Thursday bank E borrowed \$5000 from bank A. On Friday the banks balance their ledgers. If all loans were repaid, which bank would end up with \$1000 less than it had before the loans were repaid?

Organization of the facts is accomplished by using a diagram which shows where money has to be returned. Arrows indicate which bank gives money to which other bank.



Now it is a simple matter to calculate each bank's net gain or loss. We see that bank A pays out a total of \$27000 and receives a total of \$26000 so it has a net loss of \$1000.

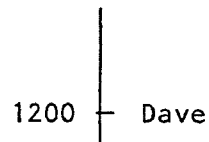
In constructing this diagram, students might want to point arrows the other way, showing where money was loaned, not where it should be returned. Emphasize that the situation we are diagramming is that where the banks return the money, not lend it.

Diagrams can often be used to help students clarify numerical order. Consider this problem:

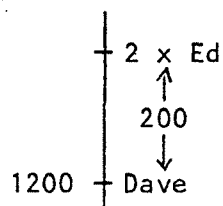
Dave's monthly income is \$200 less than double  
Ed's monthly income. If Dave makes \$1200 a month,  
what does Ed make?

Beginners often focus on the words "less than" and immediately decide to subtract. So they subtract \$200 from \$1200 to get \$1000, divide that in half, and conclude that Ed makes \$500 a month. If they draw a diagram, it will help them realize exactly which income is less than

which. We draw a line and place Dave's income on it.



Dave's income is less than twice Ed's income, so on the diagram Dave's income should lie below two times Ed's income. Dave's income and twice Ed's income are separated by \$200.

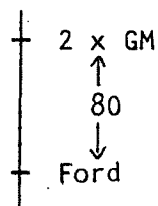


Now from looking at the diagram, it is clear that two times Ed's income is \$1400, so Ed's monthly income must be \$700.

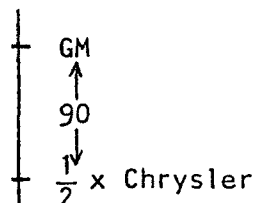
A more complicated problem which can be solved with the same technique is:

If the weekly income of an assembly line worker at General Motors doubled, he would be making \$80 a week more than a similar worker at Ford. The General Motors worker's weekly income is \$90 more than one-half of the weekly salary of a worker at Chrysler. The Chrysler worker makes \$220 a week. How much do the workers at General Motors and Ford make?

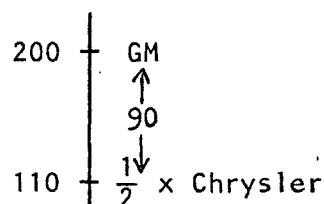
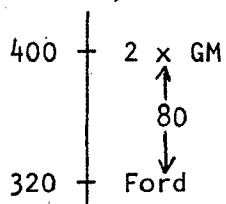
The first sentence tells us that two times the weekly income of a worker at General Motors is \$80 above a Ford worker's weekly income.



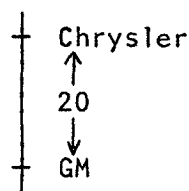
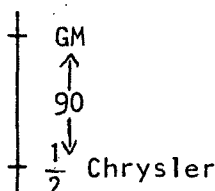
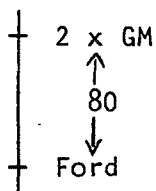
The second sentence compares General Motors' and Chrysler's salaries. Since we do not know where Chrysler would fit on the diagram we have just drawn, we make a new one to illustrate the second sentence.



Now the third sentence allows us to start putting numbers on the diagrams. Since a worker at Chrysler makes \$220 a week, one-half of his salary is \$110. A General Motors worker is \$90 above that, so he makes \$200 a week. Then twice the General Motors income is \$400 and the Ford income is \$80 below that, or \$320 a week.

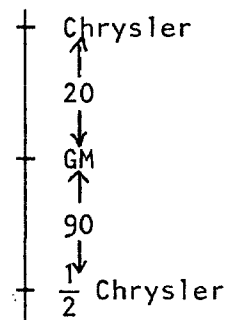


Suppose that instead of giving the Chrysler worker's salary outright, the problem states that the Chrysler worker makes \$20 a week more than the General Motors worker. The following diagrams can then be drawn:



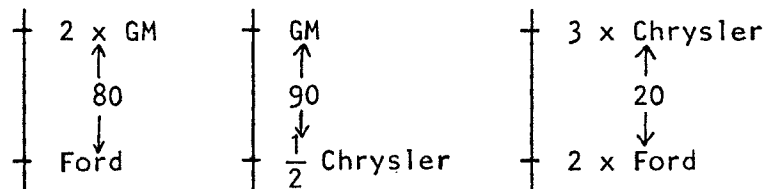
The last two diagrams can be combined.





We can then infer that since the difference between a full Chrysler salary and half a Chrysler salary is \$110, the Chrysler salary must be \$220 a week.

Now vary the problem again and suppose that the third sentence reads: Twice the Ford worker's salary is \$20 less than triple the salary of the Chrysler worker. Then the diagrams look like this:



Now it is difficult to combine the diagrams, because there are no common quantities between them. In this situation, it is easier to use the diagrams to write three equations:

$$F + 80 = 2G$$

$$\frac{1}{2}C + 90 = G$$

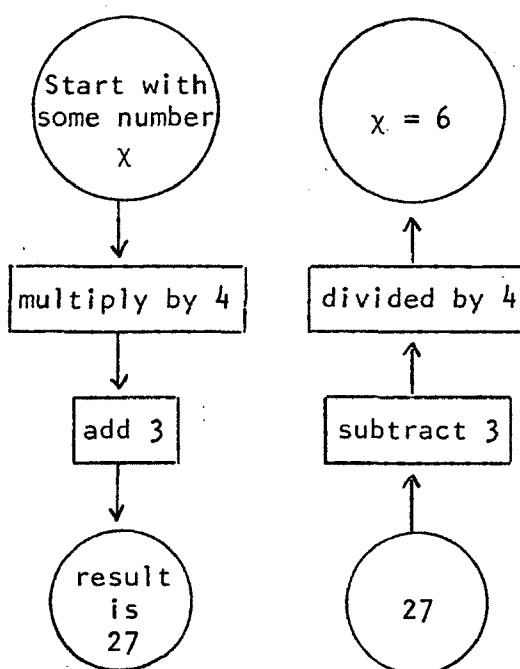
$$2F + 20 = 3C$$

This system of equations can then be solved to give numerical results.

This style of diagramming is used in Arthur Whimbey and Jack Lockhead, Problem Solving and Comprehension (Philadelphia: Franklin Institute Press, 1979). Many more problems similar to the last two may be found in this source.

Flow charting is a diagramming method which has expanded from the computer field into numerous other areas. It is a good tool for forcing a student to see a problem as being composed of numerous individual steps which are linked together in a specific sequence. An equation can be represented by a flow chart. To solve the equation, the student must reverse the flowchart, replacing each arithmetic instruction with its inverse operation.

$$4x + 3 = 27:$$



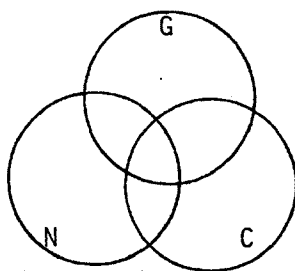
Teachers interested in further examples and uses of flowcharting in teaching beginning algebra are referred to Max A. Sobel and Evan M. Maletsky, Essentials of Mathematics, vol. 3: Skills and Concepts (Boston: Ginn & Co., 1969).

Venn diagrams are often very useful in helping to sort out numerical information. Many upper elementary mathematics textbooks contain problems which can be solved using Venn diagrams, but the procedure is seldom used in high school math classes. Consider this problem:

In a survey of 70 voters, 12 indicated they supported legalized gambling, 18 were in favor of more nuclear power plants, and 24 supported increased coal production. Of these, 6 supported both legalized gambling and nuclear power plants, 4 were for legalized gambling and increased coal production, and 10 were in favor of nuclear power plants and increased coal production. One person was in favor of all three issues.

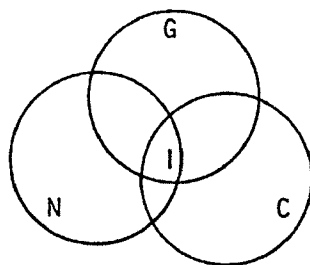
- (a) How many were in favor of none of the items?
- (b) How many were only in favor of legalized gambling?
- (c) How many supported only nuclear power plants?
- (d) How many supported only increased coal production?

When presenting this problem to a class, it is a good idea to let them try answering the questions as a group with no initial direction from the teacher. They soon decide that the overlapping information is very confusing, and they need some scheme for sorting it out. The teacher could suggest that they imagine inviting all 70 people out to the football field and try to separate them so that all the people who supported an issue were standing in a common area. We could represent the area where we want the people supporting legalized gambling to stand by a circle. The students are quick to realize that if we want to represent the other two issues by circles also, then the three circles must overlap. So we start with the following diagram:

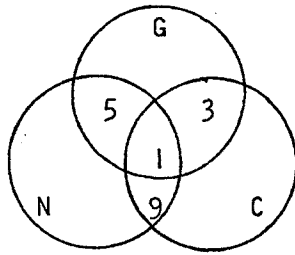


G signifies the circle where the supporters of legalized gambling will stand, N is the circle for those in favor of nuclear power plants, and C is for the supporters of increased coal production.

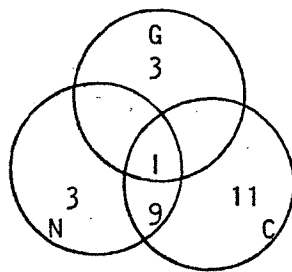
Notice that each circle is divided into four sections. We cannot just ask all those who favor nuclear plants to stand in the N circle; it makes a difference in which section they stand. Ask the class who is the easiest person to place, and they should realize it is the one person who is in favor of all three issues. He must stand in  $G \cap N \cap C$ . (Be cautious about using this notation if the class is not comfortable with it.)



Now consider those people who stand in the overlap of two circles. There should be a total of 6 people in  $G \cap N$ . One person is already there, so we must have 5 people in  $G \cap N$  but not in  $G \cap N \cap C$ . Similarly, we need 3 in  $G \cap C$  but not in  $G \cap N \cap C$ , and there must be 9 people in  $N \cap C$  but not in  $G \cap N \cap C$ .



We are now ready to place those people who are in favor of only one issue. Twelve people stand in the G circle; 9 are already there so we must add 3 more in the section not already filled. We also must add 3 to the N circle and 11 to the C circle. The completed diagram is shown.



Now the questions are easy to answer. We see that there are a total of 35 people standing in the circles. Since 70 people were surveyed, 35 must be standing outside the circles, signifying that they were in favor of none of the issues. Three people were in favor of legalized gambling only, three supported just nuclear power plants, and 11 were only in favor of increased coal production.

The problems in this chapter were specifically chosen to illustrate the value of drawing diagrams to help clarify a given situation. This is a basic problem solving strategy, and students should be encouraged to use it whenever it seems appropriate. Following are some additional problems to give the students more practice with the techniques just introduced.

## PROBLEMS

1. One day last January, Gary, Nancy, Rick, and Diane went ice fishing at Georgetown Lake. Gary and Nancy together caught 12 more fish than twice the number Diane caught. Rick, who caught 17 fish, had 4 less than Nancy but 6 more than Diane. How many fish did Gary catch?
2. Marsha makes \$25 a week less than the sum of what Judy and Gail together make. Gail's weekly income would be triple Carol's if she made \$50 more a week. Carol makes \$75 a week. How much more than Judy does Marsha make?
3. In a survey of 25 college students, it was found that of the three Montana newspapers, Missoulian, Great Falls Tribune, and Billings Gazette, 12 read the Missoulian, 11 read the Tribune, 10 read the Gazette, 4 read the Missoulian and the Tribune, 3 read the Missoulian and Gazette, 3 read the Tribune and Gazette, and one person read all three.
  - (a) How many read none of the newspapers?
  - (b) How many read the Missoulian alone?
  - (c) How many read the Tribune alone?
  - (d) How many read the Gazette alone?
  - (e) How many read neither the Missoulian nor the Tribune?
  - (f) How many read the Missoulian or the Tribune or both?

## ANSWERS

1. Gary caught 13 fish.

2. Marsha makes \$150 a week more than Judy.

3. (a) 1 (b) 6 (c) 5 (d) 5 (e) 6 (f) 19

## USING MANY VARIABLES

A great many problems require students to translate English phrases into proper algebraic relationships. The ability to translate correctly is a fundamental necessity for successful problem solving. If the problem is translated incorrectly, all the sophisticated mathematics and clever strategies in the world will not yield a valid solution. There is a simple technique which can help students translate word problems correctly, and that is the use of several variables. This technique is taught during the study of simultaneous equations, but in the author's opinion its use is often neglected in other areas of the curriculum. Most high school algebra textbooks will solve problems using the minimum number of variables, but the relationships in a problem and the algebraic translations of those relationships can often be made much clearer by using more variables. Here is a typical problem which would appear quite early in an Algebra I text. A traditional one-variable solution is given and also a multi-variable solution.

Joe, Harry, and Sue all collected cans for recycling.

Joe collected 29 pounds more than Harry but 13 pounds less than Sue. Three times the weight Joe collected is 43 pounds less than twice the weight Harry collected added to twice the weight Sue collected. How many pounds of cans did each one collect?



The one-variable approach requires that we first assign a variable to represent the number of pounds that one person collected. Immediately the students are faced with a problem other than that of translation. The question asks for the weight of cans that three people collected, but we can only use one variable. What should it represent? The ability to choose efficient variable assignments is a skill that comes with experience, and sometimes those choices are very difficult to justify to a student without such experience.

Suppose it is decided to let  $x$  represent the number of pounds that Harry collected. We are then faced with the task of writing expressions for the number of pounds that Joe and Sue collected. The expression  $(x + 29)$  for the number of pounds that Joe collected is fairly obvious to students. To write an expression for the number of pounds Sue collected, they must realize that she collected 13 pounds more than Joe. The number of pounds she collected is then  $(x + 29) + 13 = x + 42$ . It is very confusing for some students to see the words "13 pounds less than" translated into "+ 13" in the expression. Use of vertical diagrams as introduced in the previous chapter might help clarify the reasoning. The equation to solve the problem is then translated as

$$3(x + 29) = 2x + 2(x + 42) - 43.$$

Understanding all the steps that lead to the final equation is very difficult for the average student. In addition to deciding what the variable should represent, he has to sort through the problem to decide which information leads to an equation and which gives expressions for the other unknowns. These are valuable skills to learn, but the whole

process is a little overwhelming to students who are just beginning to translate words into equations.

Once students become comfortable with the concept of a variable, they seem to want to assign a variable to every unknown quantity. Why not let them follow their instincts and use as many variables as they think are necessary? For the above problem we could use:

$J$  = number of pounds Joe collected

$H$  = number of pounds Harry collected

$S$  = number of pounds Sue collected

It is then fairly straightforward to arrive at the following equations:

$$J = H + 29$$

$$J = S - 13$$

$$3J = 2H + 2S - 43$$

In writing these equations the emphasis is on translation, which is what the student is supposed to be practicing at this point. It makes the translation much easier if the students choose letters for the variables that are related to the quantities they represent. To solve the equations, the teacher must show the students the substitution technique. There is really no reason why this method cannot be introduced before graphs, slopes, and simultaneous equations are studied. Students accept the technique quite readily, and it creates a good opportunity for stressing the meaning of equality and equivalent equations.

Here is another example of a typical Algebra I problem that lends itself to a multi-variable solution:

Joan scored 58 points on a test, receiving five points for each correct answer and losing two points for each incorrect answer. If the number she had correct was two more than twice the number she had incorrect, how many did she have correct?

A one-variable solution would have one of the following two forms:

$x$  = number of correct answers

$\frac{1}{2}(x - 2)$  = number of incorrect answers

equation:  $5x - 2\left[\frac{1}{2}(x - 2)\right] = 58$

or

$x$  = number of incorrect answers

$2x + 2$  = number of correct answers

equation:  $5(2x + 2) - 2x = 58$

In the first version, it is extremely difficult for most students to arrive at an expression for the number of incorrect answers. The second version requires assigning the variable to a quantity not being sought, which can also be confusing for beginners.

Now consider a solution using more than one variable.

$c$  = number of correct answers

$i$  = number of incorrect answers

equations:  $c = 2 + 2i$

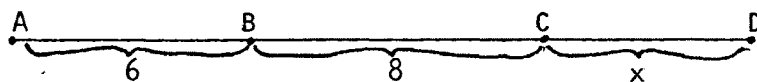
$5c - 2i = 58$

The translations in this case are very straightforward and the resulting equations are very easy to solve by substitution.

As students gain experience with word problems, they will need to rely on a many-variable strategy less and less. They can always return to it, however, to help analyze a particularly confusing problem. Consider the following advanced problem:

A treasure is located at a point along a straight road with towns A, B, C, and D on it, in that order. A map gives the following instructions for locating the treasure: (1) Start at town A and go half of the way to C. (2) Then go one-third of the way to D. (3) Then go one-fourth of the way to B, and dig for the treasure. If  $AB = 6$  miles,  $BC = 8$  miles, and the treasure is buried midway between A and D, find the distance from C to D.<sup>1</sup>

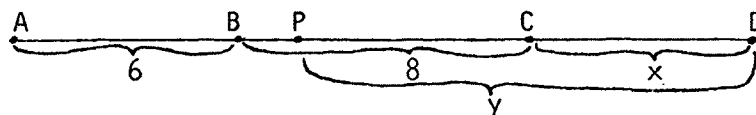
The solution requires analyzing each instruction in turn. A picture of the road and town suggests a number line, so let us use that as a model. It would be convenient to have a reference point; for simplicity choose point A to be zero.



We do not know  $CD$  so let the first unknown,  $x$ , be the distance from C to D. Step 1 says to start at A and go half of the way to C. The distance from A to C is 14, so after step 1 we are at the point P, whose coordinate is 7.

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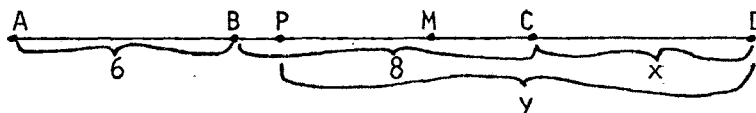
<sup>1</sup>Loren Johnson, et. al., How to Boil an Egg in 15 Minutes and Other Problem Solving Exercises, Missoula, 1977, p. 69. (Mimeographed.)



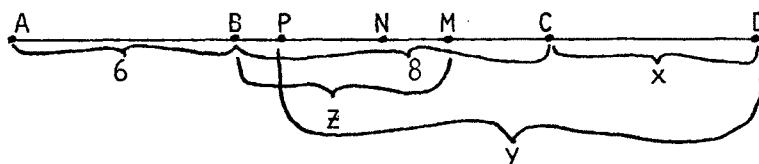
Step 2 requires us to go one-third of the distance from P to D.

Since we do not know this distance, let  $y$  be the distance from P to D.

By studying the drawing, we see that  $y = 7 + x$ . We then go  $\frac{1}{3}y$ , and end up at some point M, whose coordinate is  $m$ . Then  $m = 7 + \frac{1}{3}y$ .



Now step 3 talks about the distance from M to B, again an unknown quantity. So let  $z$  be the distance from M to B, and we know that  $z = m - 6$ . If we go one-fourth of the distance from M to B, we will end up at point N, whose coordinate is  $n = m - \frac{1}{4}z$ .



We also know that N is halfway between A and D, so  $n = \frac{1}{2}(6 + 8 + x)$ .

We now have two expressions for  $n$ , which we can equate. We can use several substitutions to work back to an equation with only one unknown.

$$\begin{aligned}
 \frac{1}{2}(6 + 8 + x) &= m - \frac{1}{4}z \\
 \frac{1}{2}(14 + x) &= m - \frac{1}{4}(m - 6) \quad (\text{because } z = m - 6) \\
 \frac{1}{2}(14 + x) &= \frac{3}{4}m + \frac{3}{2} \\
 \frac{1}{2}(14 + x) &= \frac{3}{4}\left(7 + \frac{1}{3}y\right) + \frac{3}{2} \quad (\text{because } m = 7 + \frac{1}{3}y) \\
 \frac{1}{2}(14 + x) &= \frac{27}{4} + \frac{1}{4}(7 + x) \quad (\text{because } y = 7 + x) \\
 x &= 6 \text{ miles}
 \end{aligned}$$

By careful substitution and simplifying, we were able to avoid an unwieldy equation at any step. By contrast, the equation derived when

using just the variable  $x$  is  $7 + \frac{1}{3}(x + 7) - \frac{1}{4}[\frac{1}{3}(x + 7) + 1] = \frac{x + 14}{2}$ .

This equation offers numerous opportunities for errors, both in forming the equation and in solving it.

The technique of using several variables and forming several equations is certainly not unknown to students, but it is vastly under-used. Too often, it is introduced only in the chapter on simultaneous equations. Students should be exposed to the power of this technique and be encouraged to use this method whenever it seems natural or would seem to clarify a problem. At times, the solution may not be as elegant as a single variable one, but if it aids the student's understanding of a problem, then it is worthwhile. It provides the student with one more tool to use in attacking a perplexing problem. Following are some problems which lend themselves to multi-variable solutions.

## PROBLEMS

1. The sum of the measures of the three angles in a triangle is  $180^\circ$ . The first angle is  $22^\circ$  more than twice the second angle, and the third angle is  $16^\circ$  larger than the second angle. Find the measure of each angle.
2. In a salvage operation on a sunken Spanish galleon off the Florida coast, 48 artifacts were recovered. There were two fewer jeweled bracelets than carved marble figurines recovered. The number of Spanish gold pieces reclaimed was two less than twice the number of marble pieces. How many of each type of artifact were recovered?
3. The Montana Fish and Game Commissioner was to be in Missoula for a public hearing at 5:00 in the afternoon. He was delayed in Helena and did not leave as soon as he had planned. At the time he left, he calculated that if he drove 50 miles per hour, he would reach Missoula at 5:30. If he traveled 60 miles per hour, he would arrive 20 minutes early. Can he get to Missoula in time for the meeting without breaking the 55 mile per hour speed limit?

## SOLUTIONS

1. Variables:  $A_1$  = measure of first angle  
 $A_2$  = measure of second angle  
 $A_3$  = measure of third angle  
Equations:  $A_1 + A_2 + A_3 = 180$   
 $A_1 = 22 + 2A_2$   
 $A_3 = 16 + A_2$

Answers:  $A_1 = 93^\circ$ ,  $A_2 = 35.5^\circ$ ,  $A_3 = 51.5^\circ$

2. Variables:  $B$  = number of bracelets  
 $M$  = number of marble figurines  
 $G$  = number of gold pieces

Equations:  $B + M + G = 48$

$$B = M - 2$$

$$G = 2M - 2$$

Answers:  $B = 11$ ,  $M = 13$ ,  $G = 24$

3. Variables:  $D$  = distance from Helena to Missoula  
 $T_1$  = time at 50 mph  
 $T_2$  = time at 60 mph  
 $R$  = rate at which he would reach Missoula June at 5:00  
 $T$  = time it takes traveling at rate  $R$

Equations:  $D = 50T_1$

$$D = 60T_2$$

$$D = RT$$

$$T = T_1 - \frac{1}{2}$$

$$T = T_2 + \frac{1}{3}$$

Answers:  $D = 250$  miles

$$T_1 = 5 \text{ hours}$$

$$T_2 = 4\frac{1}{6} \text{ hours}$$

$$R = 55\frac{5}{9} \text{ mph}$$

$$T = 4\frac{1}{2} \text{ hours}$$

He cannot get to Missoula on time without going slightly over the speed limit.

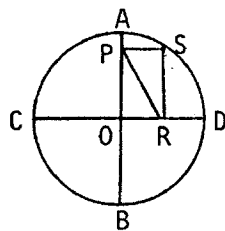


## CHANGING PERSPECTIVES

Everyone has experienced the frustration of struggling with what we thought was a difficult problem and then being shown a very simple solution. Usually the solution in these cases seems to be based on some sort of clever trick, and our initial reaction is "How did they ever think of that?" Students also experience that feeling, and too often they are left with only the solution to the problem and not with any indication of how the solution was discovered. While the ability to reach such solutions increases with experience, students must be given some tools to help them along. Otherwise it is very difficult to gain the necessary experience.

One helpful technique is to teach students to look at a problem from more than one perspective. This article contains six problems that can be viewed in more than one way. Two of them cannot be solved unless they are viewed from the proper perspective. The other four can be solved directly, but the solution becomes much simpler with a change of perspective.

The first problem is one that might be found in any elementary geometry book.



Circle  $O$  has a radius of 10 cm.  $\overline{AB}$  and  $\overline{CD}$  are perpendicular diameters,  $\overline{SP} \perp \overline{OA}$  and  $\overline{SR} \perp \overline{OD}$ . Find the length of  $\overline{PR}$ .

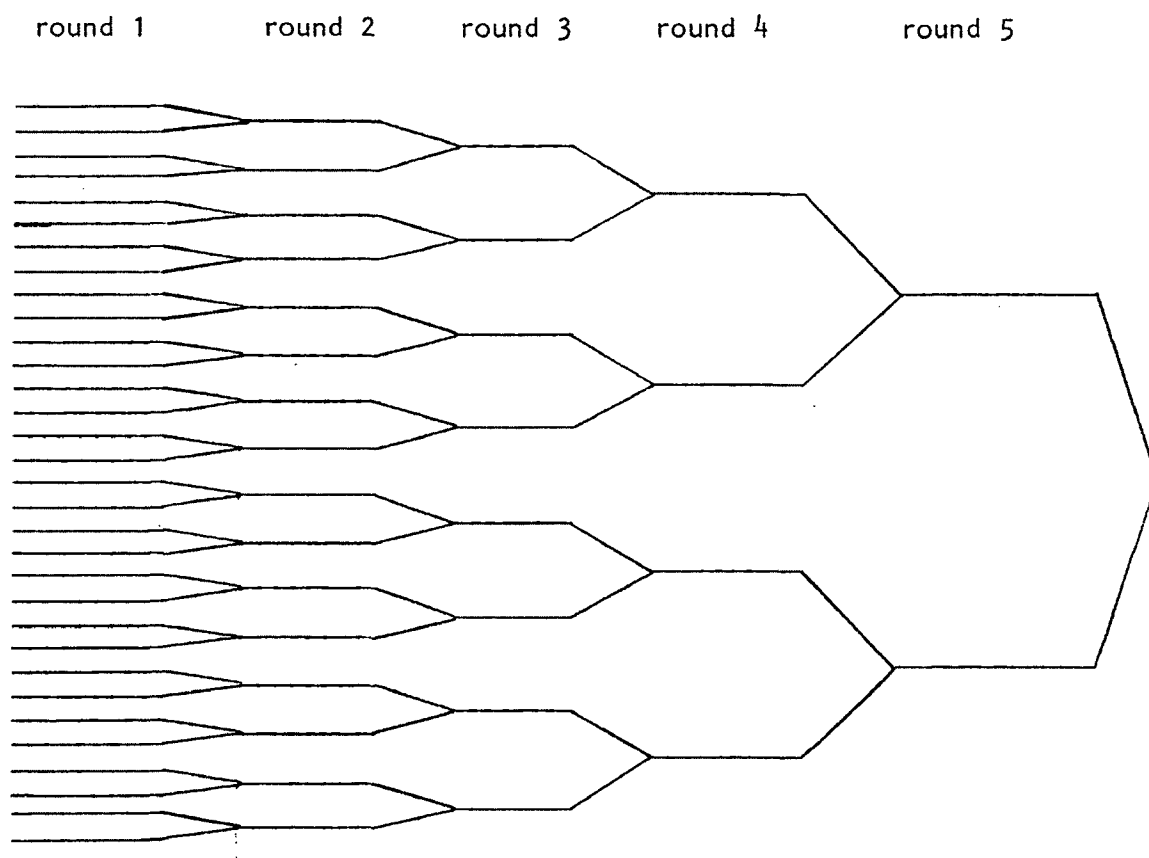
Most students immediately focus on  $\triangle OPR$  and attempt to find the length of  $\overline{PR}$  by using the Pythagorean Theorem. They quickly discover that there is not enough information given to solve the problem this way. The solution is found by viewing  $\overline{PR}$  as part of another geometrical figure, rectangle  $OPSR$ .  $\overline{PR}$  is a diagonal of this rectangle, so its length must be the same as the length of the other diagonal,  $\overline{OS}$ . However,  $\overline{OS}$  is a radius of the circle so its length is immediately known to be 10 cm.

Notice how the problem is worded to obscure the proper perspective.  $OPSR$  is never referred to as a rectangle, and  $\overline{OS}$  is not drawn. Textbook authors do this in an attempt to prevent students from solving problems by rote. However, this practice can easily lead to frustration on the students' part if they are not acquainted with the strategy of changing perspectives. Many students do not know how to modify their approach if their first attempt does not lead to a solution. Changing perspective is not always the modification that is needed, but students should be aware that it is one of the possible approaches to try.

Sometimes a problem can be solved directly, but a change of perspective will help find alternate methods of solution. Here is an example of a problem whose direct solution is rather tedious. Changing perspective leads to a much more elegant solution.

Thirty-two players are entered in a single elimination tennis tournament. What is the total number of matches that must be played to determine a champion?

This problem can be solved by drawing an elimination diagram. Each line represents a player, and joined lines indicate a match. Only the winner of each match advances to the next round.



We then simply count the number of matches in each round to get  $16 + 8 + 4 + 2 + 1 = 31$  total matches.

The same answer could also be reached without the diagram by realizing that the number of matches in each round is half the number of players in that round, and each round eliminates half of the remain-

ing players. In the first round there are 32 players and 16 matches, in the second there are 16 players and 8 matches, and so on.

These solutions concentrated on the number of winners in each round. Now change perspective and concentrate on the losers. Each match eliminates one player. With 32 players, 31 must be eliminated to determine a champion. So 31 matches must be played. This approach gives the answer much faster than the first one. The first perspective leads to an even more complicated solution if the number of players is not a power of two; in that case some players will have a bye in some matches. The second method generalizes easily to any number of players.

Sometimes mathematical experience itself can block the easiest approach to a problem. Here is a problem that pre-algebra students will solve easily but that advanced algebra students will usually find quite difficult.

A bin contains 250 bushels of winter wheat. A higher protein content is desired so a certain amount of winter wheat is removed and replaced by the same amount of spring wheat. After the spring wheat is mixed in, it is decided to raise the protein content even more so the same amount of mixture is removed and is replaced by the same amount of spring wheat. The final mixture contains winter wheat and spring wheat in the ratio of 16:9, respectively. How much winter wheat was removed in all?

Students who have learned to solve problems by breaking them into pieces and then attacking each piece will try to determine how much winter wheat was removed each time. It is possible to solve the problem this way, but the equation is rather complicated to set up and results in a quadratic expression. The more naive student will simply focus on the final mixture. If  $x$  bushels of winter wheat were removed, then  $x$  bushels of spring wheat must have been added, and  $250 - x$  bushels of winter wheat remain. So we have the proportion  $\frac{250 - x}{x} = \frac{16}{9}$ , which gives  $x = 90$  bushels. It is possible to solve this problem without the use of a variable at all by noting that the amount of winter wheat removed is equal to the amount of spring wheat in the final mixture. The final mixture is  $\frac{9}{25}$  spring wheat and  $\frac{9}{25}(250) = 90$ .

In this case the change of perspective was accomplished by concentrating on the end result of a process rather than on the process itself. In this problem that change simplified the solution; in others it will not. Students should be aware of both perspectives so they can choose the appropriate one for a particular situation.

A problem which cannot be solved without a change of perspective is this:

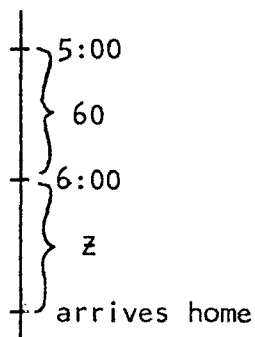
A commuter is in the habit of arriving at his suburban station each evening at exactly 6:00 P.M. His wife is always waiting for him with the car; she too arrives at exactly 6:00 P.M. She never varies her route or her rate of speed. One day he takes an earlier train, arriving at the station at 5:00 P.M. He decides not

to call his wife, but begins to walk home along the route she always takes. They meet somewhere along the route, he gets into the car, and they drive home, arriving 10 minutes earlier than usual. How long had he been walking when he was picked up by his wife?<sup>1</sup>

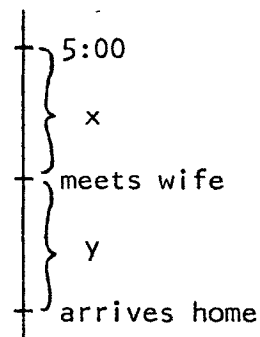
A student who tries to solve this problem by considering only the husband will find there is not enough information. By changing perspective to consider how long the wife travels, the problem becomes very simple. Since they arrive home 10 minutes earlier than usual, she travels 5 minutes less in each direction. Therefore she meets him 5 minutes earlier than usual, or at 5:55. He must have been walking for 55 minutes.

Some students want to write variables and form equations for this problem. We can draw a diagram and use several variables to help us. Let  $x$  be the number of minutes he walks,  $y$  be the number of minutes he rides, and  $z$  be the number of minutes it takes to ride all the way from the station.

usual day



this day




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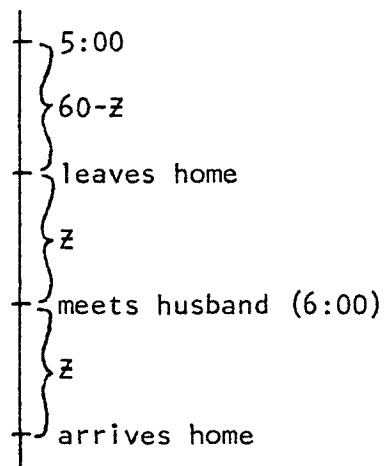
<sup>1</sup>Problem taken from Stephen Krulik and Jesse A. Rudnick, Problem Solving, A Handbook for Teachers (Boston: Allyn & Bacon, 1980), p. 103.

On the day in question, the commuter arrives home  $x + y$  minutes past 5:00. On an ordinary day he arrives home  $z$  minutes past 6:00, or  $x + 60$  minutes past 5:00. Since he arrives home 10 minutes earlier than on an ordinary day, we can write the equation:

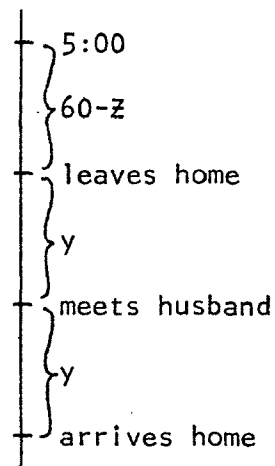
$$(1) \quad x + y = 60 + z - 10$$

Before we can solve this, however, we need some more information about the variables. We can get this information by switching our perspective to the wife's point of view.

usual day:



this day:



On the day in question, she leaves the house, as she usually does,  $z$  minutes before 6:00, or  $60 - z$  minutes after 5:00. She travels for  $y$  minutes to meet her husband, and then travels  $y$  minutes back home. So she arrives home at  $(60 - z) + 2y$  minutes after 5:00. This is 10 minutes before her usual arrival time of  $60 + z$  minutes past 5:00, so we have the equation  $(60 - z) + 2y = (60 + z) - 10$ . This equation gives the relation  $y = z - 5$ . Substituting for  $y$  in equation (1), we find that  $x = 55$  minutes.

In the previous problem, a change of perspective was needed to uncover enough information to complete the problem. In the following example, there is enough information to solve the problem directly, but a change of perspective shortens the solution a great deal.

Two bicycle riders, Kate and Susie, are 25 miles apart, riding towards each other at speeds of 15 miles per hour and 10 miles per hour, respectively. A fly starts from Kate and flies toward Susie, then back to Kate and so on. The fly continues flying back and forth at a constant rate of 30 miles per hour, until the bicycle riders meet. How far has the fly traveled?<sup>2</sup>

The question is how far the fly has traveled, so the most direct approach is to try and calculate that distance. The fly flies several "laps" between Kate and Susie, and its total distance is the sum of those laps. While the fly is flying the first lap, Susie travels a distance of  $s_1$  and Kate travels a distance of  $k_1$ . All these distances are traveled in the same amount of time, so we can use the relation  $T = \frac{D}{R}$  to set up some proportions:  $\frac{25 - s_1}{30} = \frac{s_1}{10} = \frac{k_1}{15}$ . Solving, we find that  $s_1 = 6.25$  miles and  $k_1 = 9.375$  miles. So the fly travels  $25 - 6.25 = 18.75$  miles on the first lap and the girls are then  $25 - 6.25 - 9.375 = 9.375$  miles apart. We can now calculate how far the fly travels on the second lap using the proportions  $\frac{9.375 - k_2}{30} = \frac{s_2}{10} = \frac{k_2}{15}$ . We can continue in this manner until the girls "meet." Obviously, some of the answers

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<sup>2</sup> Ibid., p. 105.



will have to be rounded off, and the number of decimal places retained will determine how many laps can be calculated. Using an accuracy of three decimal places, eleven laps can be calculated, and the total distance the fly travels is 29.989 miles. Students should realize that this is an approximate answer because of the rounding.

The method just described, while straightforward, is long, tedious, and necessarily involves some round-off error. There is a much simpler solution if we focus not on the total distance the fly travels but on the total time it flies. It flies back and forth for the length of time it takes Kate and Susie to meet. Kate and Susie start out 25 miles apart and move toward each other at a speed of 25 miles per hour, so they will meet in 1 hour. Thus, the fly flies for 1 hour at 30 miles per hour, so it must travel 30 miles.

When solving this problem, students should realize that they will have to use the relationship  $D = RT$  in some manner. Since we already know the fly's rate, finding either its distance or its time will allow us to answer the question. Finding the distance is quite cumbersome, so students should consider finding the time instead.

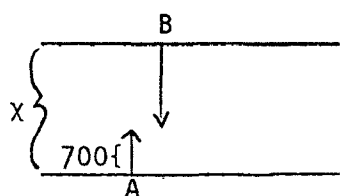
Here is a similar example which could be used as an exercise after the previous example has been discussed.

Two ferry boats ply back and forth across a river with constant speeds, turning at the banks without loss of time. They leave opposite shores at the same instant, meet for the first time 700 feet from one shore, continue on their way to the banks, return

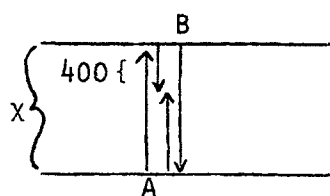
and meet for the second time 400 feet from the opposite shore. Find the width of the river.<sup>3</sup>

Several variations of a direct approach are possible, but with any approach a diagram will help clarify matters. Label the two ferry boats A and B, as indicated.

first meeting:



second meeting:



Let  $D_A$ ,  $R_A$  and  $T_A$  refer to the distance, rate, and time of ferry A and  $D_B$ ,  $R_B$ ,  $T_B$  represent those of ferry B. Then we have  $\frac{D_A}{D_B} = \frac{R_A T_A}{R_B T_B}$ . The crucial thing to realize in any solution is that when they meet, the two ferries have been traveling for the same time. Let  $D_{A_1}$  and  $D_{B_1}$  be the distances the two boats have traveled at their first meeting and  $D_{A_2}$  and  $D_{B_2}$  be their distances at their second meeting. Then since  $T_A = T_B$  at the two meetings, we have that  $\frac{D_{A_1}}{D_{B_1}} = \frac{R_A}{R_B} = \frac{D_{A_2}}{D_{B_2}}$ , and since  $R_A$  and  $R_B$  are constant,  $\frac{D_{A_1}}{D_{B_1}} = \frac{D_{A_2}}{D_{B_2}}$ . Now if we let  $x$  be the width of the river, by referring to the diagram we see that:  $D_{A_1} = 700$ ,  $D_{B_1} = x + 400$ , and  $D_{B_2} = x + (x - 400) = 2x - 400$ . We can solve the proportion  $\frac{700}{x + 400} = \frac{x + 400}{2x - 400}$  to get  $x = 1700$  feet.

This solution is certainly within reach of most high school students, although they would classify it as a difficult problem. An alternate approach is suggested by carefully studying the diagrams. Notice that at

<sup>3</sup>Ross Honsberger, Dolciani Mathematical Expositions, no. 3: Mathematical Morsels (n.p.: Mathematical Association of America, 1978) p. 7.

the first meeting the sum of the boats' distances is equal to the width of the river. (We already used this information when we wrote  $D_{B_1}$ , as  $x - 700$ .) Furthermore, at the second meeting, the sum of the distances is three times the width of the river. This observation gives us some added information about the times of the two meetings. Since the rates remained constant, and the total distance traveled was three times farther at the second meeting, it must have taken the boats three times as long to meet the second time as it did the first time. Specifically focusing on boat A, it must have traveled three times farther by the second meeting than it had gone at the first meeting. Since boat A had gone 700 feet at the first meeting, it must have gone 2100 feet by the second meeting. A glance at the second diagram shows that  $x + 400 = 2100$ , or  $x = 1700$  feet.

In presenting this material, the teacher must be careful not to make the solutions to these problems seem to be "out-of-the-hat" tricks. The students should clearly understand that changing perspectives is a problem solving technique. It is not an easy technique to use; sometimes it is extremely difficult to abandon what seems to be the natural approach. It will not work on all problems, but it gives students something to try if their first attempt does not lead to a solution. It also gives them a tool to use in searching for alternate solutions to a problem. In the ferry boat problem, for example, the easier solution might be discovered by exploring some interesting relationships that appear in the first solution, i.e. that  $D_{A_1} + D_{B_1} = x$  and  $D_{A_2} + D_{B_2} = 3x$ .

There is no easy way to determine when a change of perspective is needed and how to choose the best perspective for a problem. These are matters which become easier with experience. A first step is to simply become aware that different perspectives can be used and are sometimes advantageous. This strategy should be specifically mentioned to students each time it is employed. They need to come in contact with several instances of its use in order to become aware of it and begin to employ it themselves. The teacher should be very careful not simply to tell the students the proper perspective for a problem. They must have the experience of discovering it themselves, perhaps with some progressive hints from the teacher. This personal experience is essential. It is suggested that the teacher do a few of the problems in this chapter as examples and the others be tried by the students with teacher direction as needed.

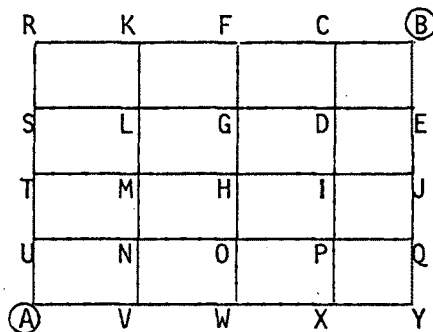
## BACK TO FRONT

Many problems which involve a sequence of decisions appear to be overwhelmingly complex. It seems impossible to sort through all the options to choose the best one for reaching the final goal. These problems often yield to the technique of back to front analysis. That is, start at the final goal and work back through the decisions one at a time to see the best way to arrive at each one. This approach could also be thought of as working a simpler problem. We assume we only have to make the very last decision, and ask how we would make that one. Then we analyze how we would make the next to last decision, and so on. We work our way back through the decisions until we have reached the beginning.

To introduce this technique to a class, the teacher could play this simple game with one of the students. Player A names an integer between 1 and 10. Player B can then name any integer up to 10 more than the number player A named. For instance, say player A named 5. Player B could then name any number from 6 to 15. Player A then names any number up to 10 more than player B's number. The players alternate in this manner until one player names 100, and that player then wins the game. Students immediately suspect there is a winning strategy, and they are eager to discover it, especially if they are playing against the teacher. After a few games, they realize that the player who says 89 will win the game, because the other player then has to choose a number from 90 to 99.

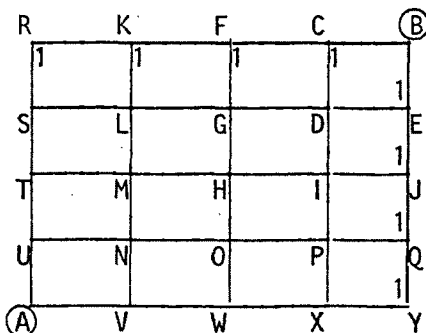
The player who said 89 will then be able to name 100 on his next turn. The problem now becomes not how to reach 100 but how to reach 89. A player can choose 89 if he forces the other player to name a number from 79 to 88. He can do this by saying 78. Similarly, he is guaranteed to reach 78 if he says 67. Students should be able to follow this sequence of winning numbers all the way back to discover that the first player can always win if he starts with 1.

Now apply this approach to the problem of determining how many different routes there are from point A to point B on the following map.

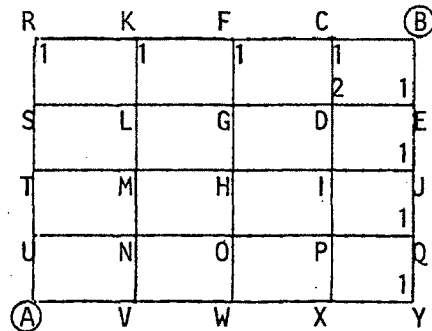


We shall make the practical assumption that we always move toward the destination at B, so we restrict ourselves to moving only north and east.

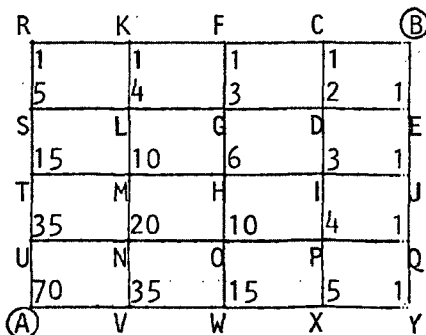
Instead of considering the problem of moving from A to B, consider the problem of going from C to B. There is obviously only one route. Similarly, there is only one route from F, K, R, E, J, Q, or Y to B. We put a small 1 by each of these intersections to help remember this.



Now consider the problem of traveling from D to B. We can go to C and take the one route from there to B, or we can go to E and take the one route to B. So there are two possible routes from D.

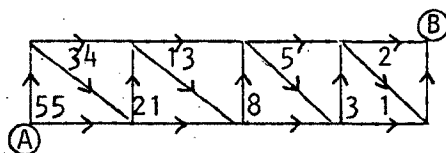


Keep moving back, one intersection at a time. From G, we could choose to go to F and take the one route from F to B, or we could go to D and have two routes from D to B. Therefore, there are three possible routes from G to B. There are also three routes from I to B. There are four routes from L, one if we choose to go to K and three if we choose to go to G. We can move back through the entire grid in this manner, writing the number of routes from each intersection to B beside the intersection. From any intersection, there are only two choices: north or east. For each of those choices we already know the number of possible routes, so to get the total number of routes from any intersection we merely add the numbers to the north and to the east. We will eventually see that there are 70 possible routes from A to B.



Notice that the numbers at the intersections form the Pascal Triangle. This is not surprising, since we derived the number for each intersection by adding the routes from the intersections to the north and to the east. Successive rows of Pascal's Triangle are derived in exactly the same manner (with a slight change of direction).

An interesting variation of this problem is shown by the grid below, where the allowable movement along each street is given by the arrows.<sup>1</sup>



The same analysis of the number of routes from each intersection gives that there are 55 possible routes from A to B. Now the numbers at the various intersections form the Fibonacci sequence, 1, 2, 3, 5, 8, 13, ..., where each number after the first two is derived by adding the previous two terms.

This back to front sort of analysis is used a great deal in practical situations. Suppose a utility company wishes to install underground cables from point A to point B on the map shown.<sup>2</sup>

<sup>1</sup>Adapted from Stephen Krulik and Jesse A. Rudnick, Problem Solving, A Handbook for Teachers (Boston: Allyn & Bacon, 1980), p. 101.

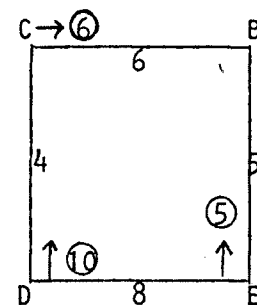
<sup>2</sup>Adapted from Engineering Concepts Curriculum Project, The Man-Made World (New York: McGraw-Hill, 1971), p. 73.



R		K		F		C		B
	3	7	5	4	3	4	6	5
S	5	6	4	8	10	8	8	7
T	4	3	7	5	9	6	5	6
U	7	7	5	3	5	3	7	8
A	6	V	6	W	4	X	5	Y

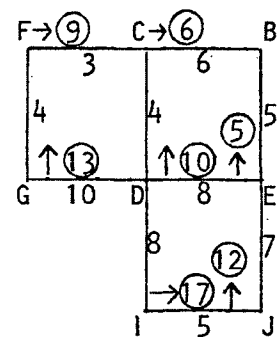
The numbers in each block represent the cost, in thousands of dollars, of installing the cables in that block. The cost varies from block to block because of the different paving materials used, the cost of providing detour signs for heavily traveled streets, and the number of underground telephone, gas, and water lines already present in the street. The company wishes to choose the route from A to B which will minimize its cost. We have already decided that on a  $4 \times 4$  grid, there are 70 possible routes from A to B. While it is possible to calculate the cost for each possible route, it certainly does not seem to be very practical. If the problem were expanded to a square twenty blocks on a side, there would be 137,846,528,820 possible routes. Calculating the cost for each possible route would be an enormous task, even for a computer.

Instead, try back to front reasoning. Consider just the block formed by intersections B, C, D, and E. If we were at C, we should have to travel east at a cost of 6 (thousand). If we were at E, our only choice would be to travel north, and the cost is 5. We indicate the direction at each intersection by an arrow

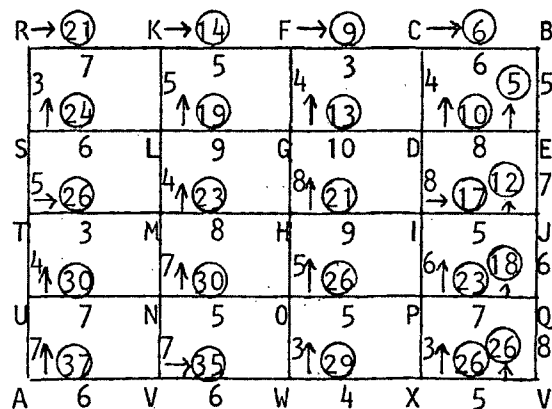


and the total cost from that intersection with a circled number. At D we have two choices: we could go to B through C at a cost of 10 or through E for a total cost of 13. The preferred direction would be to go north to C, so we indicate this with an arrow and a circled 10 for the total cost.

Now look at the next set of intersections away from B. At F we have only one choice, to go east at a total cost of 9. At J we must go north for a cost of 12. At G if we choose to go to D, it will cost 10 to get there. Then we already know that at D we should move north, and the cost from there will be 10. So the total cost if we choose to go from G to B through D will be 20. On the other hand, if we choose to go to B through F, the total cost will only be 13, so that is clearly the preferred direction. The same sort of reasoning tells us that should we find ourselves at I we would then finish the route by moving east toward J, and the least cost from that point would be 17.



We could work our way backward through the entire grid until it is completed as shown.

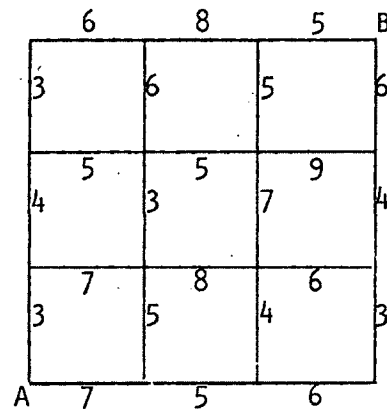


Now choosing a route is simply a matter of starting at A and following the arrow at each intersection. The preferred route is A-U-T-M-L-K-F-C-B, and the minimum cost is \$37,000.

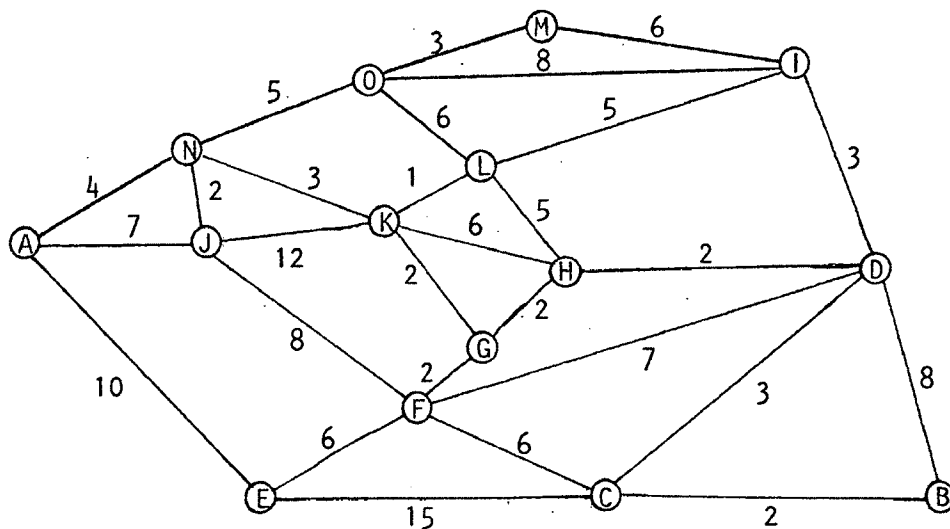
This method of analysis sometimes goes by the name of dynamic programming. It is highly adaptable to being done on a computer, although it was not originally developed for that purpose. The same method can be used to determine minimum-time routes for ambulances or fire engines, or minimum-fuel jet flight paths which take advantage of wind patterns. A discussion of dynamic programming, its history and applications can be found in The Man-Made World (see footnote 2). Examples using this technique in a probability context appear in Howard E. Reinhardt and Don O. Loftsgaarden, Elementary Probability and Statistical Reasoning (Lexington, Mass.: D.C. Heath & Co., 1977), p. 327-30.

## DYNAMIC PROGRAMMING PROBLEMS

1. In the section of city shown below how many minutes can an ambulance driver save by choosing the minimum-time route over the maximum-time route? Block times are shown.



2. In the subdivision mapped below, we wish to find the cheapest route for laying a power cable from A to B. On any block we may travel only to the right, i.e. we may travel from K to L but not from L to K. Find the cheapest route and its cost. Costs between any two points are given in thousands of dollars.



## ANSWERS

1. Minimum time is 27 minutes; maximum time is 42 minutes.  
15 minutes are saved.
2. The best route is A-N-K-G-H-D-B and the cost is \$21,000.

## THE CLOCK PROBLEM

Many classic algebra problems have fallen out of favor recently because they do not arise from practical situations. There is much to be learned from some of these problems if one's goal is not merely to arrive at a solution but to develop analytical skills. The problem presented here is one which lends itself both to generalization and to alternate methods of solution. It is very enlightening for the student to explore how the different methods of solution and the generalization are interrelated.

This is a particularly good problem to show students how to go about generalizing a solution. Often, generalization also means abstraction, and students can have a great deal of difficulty with abstract concepts. In this problem, however, the generalization still deals with concrete objects. The students can draw the situations or look at a physical model. This problem also provides a good example of how to modify a solution when needed. The initial solution is not valid for all cases of the problem. When students realize this, their first reaction may be to discard the solution all together. However, if a solution is valid some of the time, there must be some worthwhile ideas in it. In this instance, it is possible to modify the solution to account for all the cases.

A typical statement of the problem is "At what time between one and two o'clock do the hands of a clock coincide?" A straightforward method

of attack is to let  $x$  represent the number of minutes after 1:00 when the hands coincide. We need some notion of how far each hand travels in those  $x$  minutes, and the easiest approach is to consider through how many degrees each one travels. The minute hand travels  $360^\circ$  per hour, or  $6^\circ$  per minute. The hour hand travels  $30^\circ$  per hour or  $\frac{1}{2}^\circ$  per minute. So in  $x$  minutes the minute hand will travel  $6x^\circ$ , and the hour hand will travel  $\frac{1}{2}^\circ$ . Now the minute hand starts out pointing at 12, and the hour hand starts pointing at 1. So the minute hand has to travel  $30^\circ$  (the angle between 12 and 1) more than the hour hand. We can express this with the equation  $6x = 30 + \frac{1}{2}x$ , which gives  $x = 5\frac{5}{11}$  minutes. Therefore, the two hands coincide at  $1:05\frac{5}{11}$ .

One of the first things that students should be required to do upon reaching this solution is to verify that it is reasonable. We know that the hands must coincide between 1:05 and 1:10, and intuition tells us that it will happen very close to 1:05. So  $1:05\frac{5}{11}$  is certainly a realistic answer. Students should make a habit of this type of quick reasonability check. It will help them catch many errors.

To help students generalize a problem, begin by thinking of variations of the question that was asked. For instance, we could have wanted to know at what time after 5:00 the two hands would coincide. In this case the minute hand must travel  $5(30) = 150^\circ$  more than the hour hand. So the equation becomes  $6x = 150 + \frac{1}{2}x$ , and the time of coincidence is  $5:27\frac{3}{11}$ .

Another variation would be to ask at what time after 5:00 the two hands would be perpendicular. In this case the minute hand has to travel  $150^\circ + 90^\circ = 240^\circ$  farther, so the equation is  $6x = 240 + \frac{1}{2}x$ , and the

solution is  $5:43\frac{7}{11}$ . However now there are two solutions, because the hands are also perpendicular  $90^\circ$  before the minute hand catches the hour hand. Now the equation is  $6x = 60 + \frac{1}{2}x$ , and the solution is  $5:10\frac{10}{11}$ . High school students are used to problems with unique solutions, and it is important for them to realize that they must always be alert for the possibility of more than one correct answer.

Now, perhaps, the student is ready to attempt a generalization such as "How many minutes after  $c$  o'clock will the hands be  $\theta^\circ$  apart?" As before, let  $x$  be the number of minutes after  $c$  o'clock when this situation occurs. Based on his experience with the above example, a student might reason that the minute hand must be  $\theta^\circ$  ahead of the place where the hands coincide or  $\theta^\circ$  behind that place. The hands coincide when  $6x = 30c + \frac{1}{2}x$ . So they will be  $\theta^\circ$  apart when  $6x = (30c \pm \theta) + \frac{1}{2}x$ . Solving for  $x$ , we get  $x = \frac{60}{11}c \pm \frac{2}{11}\theta$ . An initial check of this equation should be to verify that it gives the correct answers for the three problems already solved. Then the student should ask if there is any possibility of the formula giving nonsensical answers. Are there any values for  $x$  that we could not accept? Since  $x$  represents the number of minutes past the hour, we must have that  $0 \leq x < 60$ . When values outside this range appear, the formula is not useful. If students have difficulty seeing that the above formula is not always valid, ask them to use it on the following problems.

1. How many minutes after 1:00 are the hands perpendicular?
2. How many minutes past 10:00 are the hands perpendicular?

The formula gives a negative result for the first problem and an answer greater than 60 for the second one.



What conditions lead to a negative answer? This occurs when

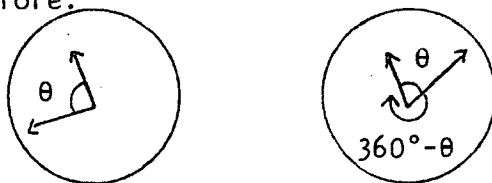
$\frac{60}{11}c - \frac{2}{11}\theta < 0$ , or when  $\theta > 30c$ . This means that on the hour the hands are already closer than  $\theta^\circ$ . However, they will be  $\theta^\circ$  apart when the minute hand is  $\theta^\circ$  past the hour hand and again when it is  $(360 - \theta)^\circ$  past.



In this case, we must modify our equations. We get:

$$\begin{aligned} 6x &= 30c + \theta + \frac{1}{2}x & \text{or} & & 6x &= 30c + (360 - \theta) + \frac{1}{2}x \\ x &= \frac{60}{11}c + \frac{2}{11}\theta & \text{or} & & x &= \frac{60}{11}c + \frac{2}{11}(360 - \theta) \end{aligned}$$

We will get an answer larger than 60 when  $\frac{60}{11}c + \frac{2}{11}\theta > 60$  or  $\frac{60}{11}c - \frac{2}{11}\theta > 60$ . These sentences give the conditions  $\theta > 330 - 30c$  or  $\theta < 30c - 330$ . A tacit assumption in this entire analysis is that 12:00 corresponds to 0. If we wanted to know how many minutes after 12:00 some angle occurred, we would have to use  $c = 0$  in the equation. After all, the minute hand would have to travel  $\frac{1}{2}x^\circ$  to catch the hour hand, not  $(30(12) + \frac{1}{2}x)^\circ$ . So the largest  $c$  can be is 11. This tells us that the condition  $\theta < 30c - 330$  is impossible.  $30c - 330 \leq 30(11) - 330 = 0$ , and  $\theta$  cannot be negative. Therefore, the only situation that will cause trouble is  $\theta > 330 - 30c$ . In this case the desired angles occur twice before the minute hand catches the hour hand, once at  $\theta^\circ$  before and once at  $(360 - \theta)^\circ$  before.



Now our equations become:

$$\begin{aligned} 6x &= 30c - \theta + \frac{1}{2}x & \text{or} & & 6x &= 30c - (360 - \theta) + \frac{1}{2}x \\ x &= \frac{60}{11}c - \frac{2}{11}\theta & \text{or} & & x &= \frac{60}{11}c - \frac{2}{11}(360 - \theta) \end{aligned}$$

The student can use the two previously given problems to check these new cases of the generalization. He should also convince himself that there are no other possibilities to be considered. We have accounted for the case when the two angles occur before the coincident time, the case when they occur after the coincident time, and the case when one occurs before and one after. Also, our formulas give a single answer for the special cases  $\theta = 0^\circ$  or  $\theta = 180^\circ$ .

In addition to generalizing a particular solution, students should be encouraged to search for different solutions to the problem. This forces the students to find other approaches to the problem and helps them practice changing perspectives. In addition, one solution often helps illuminate another one.

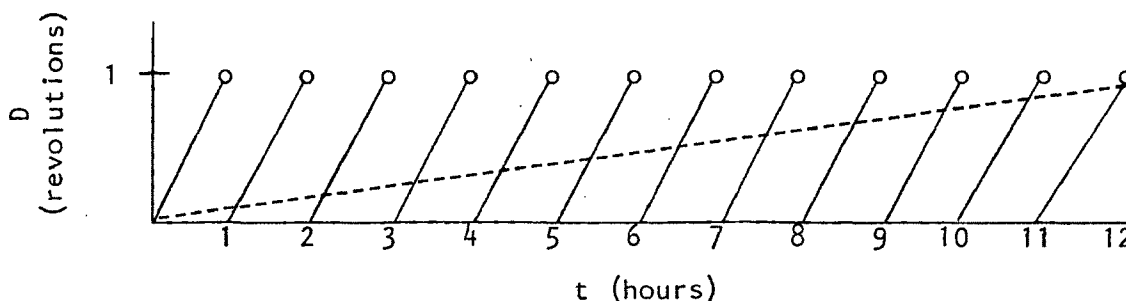
Here are two different solutions to the original problem of when the hands of a clock coincide. The first uses only arithmetic, and the second solves the problem graphically. Both solutions require that one view the problem from a completely different perspective.

The first solution comes when we forget about the face on the clock and concentrate only on the position of the hands. The hands will coincide eleven times in twelve hours. These coincident times will be equally spaced because any one of them could correspond to 12:00. So the angle between any coincident time and the next one is the same. Therefore, each interval between coincident times must be  $\frac{12}{11}$  hours or

1 hour and  $5\frac{5}{11}$  minutes. What time will the hands coincide between 5:00 and 6:00? It will be the fifth coincidence since 12:00, so the time elapsed will be  $\frac{12}{11}(5) = 5\frac{5}{11}$  hours, or 5 hours and  $27\frac{3}{11}$  minutes.

This approach also extends to some variations. The hands will form a straight angle every  $\frac{12}{11}$  hour after 6:00. They will be perpendicular every  $\frac{12}{11}$  hour after 3:00 and also every  $\frac{12}{11}$  hour after 9:00.

Another approach comes from viewing the problem as a rate problem and trying to find the instances when the minute hand overtakes the hour hand. Most high school students are familiar with the formula  $D = RT$  and should realize that if  $R$  is constant, a graph of time vs. distance will be a straight line. Any exposure to physics should acquaint them with the idea of expressing distance in revolutions and rate as revolutions per hour. If they have not had physics, this choice of units will need to be explained. The rate of the minute hand is one revolution per hour and that of the hour hand is  $\frac{1}{12}$  revolution per hour. When measuring the distance of the minute hand, we are not interested in how many complete revolutions it has made, but only in the fraction of a revolution that it has completed. What we want is to set the distance back to zero every time we complete one revolution. Now we can graph time vs. distance as shown. The graph for the minute hand is shown as a solid line, that for the hour hand as a dotted line.



If a sufficiently accurate graph were drawn, the times of coincidence could be approximated. It is probably just as easy, however, to approximate the times by looking at a clock face. The graph does serve to clarify the previous solution; it can readily be seen that the distance of the two hands will coincide eleven times at equally spaced intervals. It also serves as an example of a physically motivated non-continuous graph.

These are certainly not the only two alternate solutions to the problem, although they are two of the more unique approaches. Alternate solutions will emerge naturally from a class if the students are allowed to attempt a problem without prior direction from the teacher. When this strategy is used, it is usually less frustrating for the students if they are allowed to work in groups. It is very profitable for them to share their solutions with each other and explain how they arrived at them. Even if some students do not arrive at a solution, discovering what does not work, and why, is often as valuable as discovering what does work.

The first alternate solution was given by Neal Wagner, "The Faceless Clock," Mathematics Teacher 70 (December 1977), p. 765. The graphical solution is due to Andrejs Dunkels, "Clock Problem -- a Second Time," Mathematics Teacher 72 (May 1979), p. 322. The general solution in three cases and suggested heuristics are the work of the author.

## PROBLEMS

1. (a) How many times during an hour will the hands of a clock be perpendicular?  
(b) How many times will they be  $180^\circ$  apart?
2. What is the angle (measured clockwise) between the hands of a clock at exactly c o'clock?
3. (a) How many degrees does the minute hand of a clock move in 22 minutes?  
(b) How many degrees does the hour hand of a clock move in 22 minutes?
4. (a) Make a reasonable guess at what time between 9 and 10 o'clock the hands of the clock will be  $150^\circ$  apart.  
(b) Use the appropriate formula to find the actual times between 9 and 10 o'clock that the hands are  $150^\circ$  apart. How close was your guess?
5. At what time between 4 and 5 o'clock will the hands be  $60^\circ$  apart?  
(a) Draw a diagram of the situation the first time this happens.  
(b) Using the diagram, find the time it happens.  
(c) Repeat parts (a) and (b) for the second time it happens.

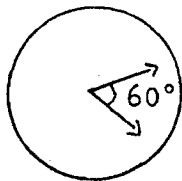
## ANSWERS

1. (a) Twice.  
(b) Once.
2. 30c degrees.
3. (a)  $132^\circ$ .  
(b)  $11^\circ$ .

4. (a) Answers vary.

(b)  $9:10\frac{10}{11}$  and  $9:21\frac{9}{11}$ .

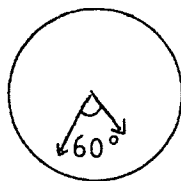
5. (a)



(b)  $4:10\frac{10}{11}$

(c)

(d)  $4:32\frac{8}{11}$



## A RECORD PROBLEM

Problems in high school mathematics classes are usually attacked with mental tools only. In developing theoretical solutions, little thought is given to the practical considerations of implementing the solution. This presentation gives students a chance to explore a problem both theoretically and by physically modeling the situation.

The problem to be solved is simply to find the length of the groove in a phonograph record. Obviously, this has something to do with the radii and circumferences of circles and so should yield to mathematical analysis. In fact, there are several theoretical solutions which will be presented later. The levels of the various solutions range from one which would be understandable to a pre-algebra student to one which requires the use of calculus.

Before such approaches are attempted, however, the students will gain much insight into what is involved if they construct a physical model of the situation. To understand the model, students should be acquainted with graphing techniques and should know how to derive the equation of a line from its graph. It is not possible with commonly available tools to measure the groove directly. Therefore, we must seek some physical substitute that will permit direct measurement. A roll of adding machine tape, ribbon, or adhesive tape would provide a reasonable model for the phonograph record that could actually be unrolled and measured.

Before actually experimenting on the roll of tape, the class should clarify what they wish to learn from the experiment. The roll of tape is composed of many layers of varying length, and the total length of the tape is the sum of the lengths of those individual layers. It would seem that if we know the number of layers and some relationship that would tell us the length of each layer, then we could calculate the total length.

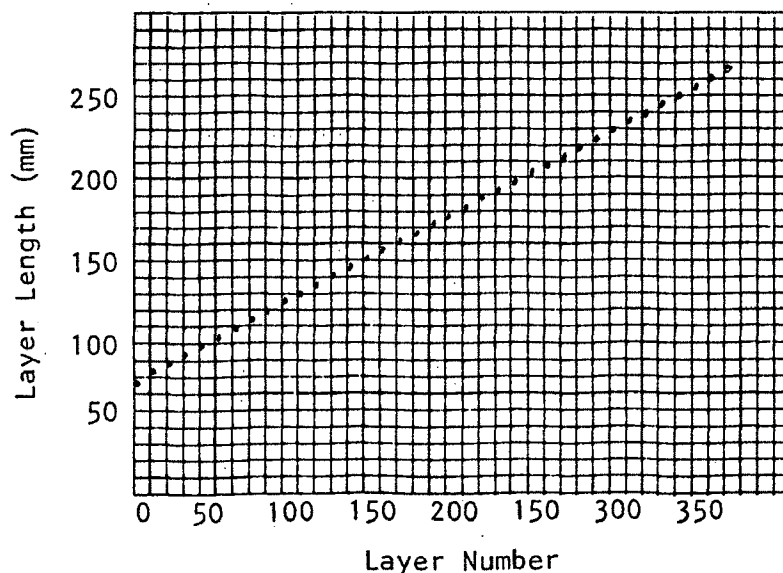
The ultimate goal is to develop a procedure that can be applied to the phonograph record problem. Thus, the final procedure should depend only on measurements that can actually be made on the record, namely the inner and outer radii of the playing surface, the speed at which the record rotates, and the length of time it plays. The advantage to using the tape, however, is that we can actually unroll it, make measurements, and thus perhaps discover some useful relationships.

In order to count the number of layers in the tape roll, it is necessary to somehow mark the beginning of each layer. One way would be to mark a radius on the end of the roll of tape and then make a cut along the radius with a sharp knife. When the tape is unrolled, each cut along the edge will mark the beginning of a new layer. The students could then measure the length of each layer and assemble their results in a chart showing layer number and length. Here is a partial chart of results, taken from Hirstein (1980). Here the layers were numbered from the innermost layer outward, and the lengths were measured in millimeters.



Layer	Length	Layer	Length	Layer	Length
1	69	130	139	250	205
10	74	140	145	260	210
20	79	150	150	270	216
30	84	160	156	280	221
40	90	170	161	290	226
50	95	180	166	300	232
60	101	190	172	310	237
70	106	200	177	320	243
80	111	210	183	330	248
90	117	220	188	340	253
100	122	230	194	350	259
110	128	240	199	360	265
120	134				

Remember that with the phonograph record we cannot measure the layers themselves, so we are searching for a relationship that will allow us to calculate the length of each layer. When faced with a mass of data, one good way to discover a relationship is to make a graph. Immediately a whole area of discussion opens on how to best construct the graph. What units shall be used? Should the layers be numbered from inside to outside or outside to inside? Do we need to graph every single layer? Which should be the independent variable and which should be the dependent? Students should be allowed to discuss these questions and make their own decisions. The above data give the following graph:



When the students construct their graphs, they should discover a linear relationship between layer length and layer number. They should be able to express this in the form  $\ell = mn + b$  where  $\ell$  is the layer length,  $n$  is the layer number, and  $m$  and  $b$  are constants calculated from the graph. A linear relationship is to be expected, since the increase in the length of the layers is due to the thickness of the paper, which is constant. Now the only remaining problem is to add the lengths of the individual layers to get the total length of the roll. If a computer or programmable calculator is available, it is a simple matter to write a short program to do this. It is interesting to compare this result with the actual measured length of the tape. There will probably be some discrepancy due to measurement errors, but it should be slight.

The length of the tape can also be found by using the formula for the sum of the first  $n$  natural numbers. If  $L$  is the total length of the tape, we have

$$\begin{aligned} L &= (m \cdot 1 + b) + (m \cdot 2 + b) + \dots + (mn + b) \\ &= m(1 + 2 + \dots + n) + bn \\ &= m \frac{n(n+1)}{2} + bn \end{aligned}$$

This equation will tell us the total length of the tape if we know  $m$ ,  $b$  and the total number of layers. We can find  $m$  and  $b$  if we know just two layers and their lengths, say the first layer and the last one.

With adding machine tape, the only way to count the number of layers was either to unroll the tape or to know the thickness of the paper. With a phonograph record we have a bit more information. We know the number of revolutions it makes per minute, and we can time the number of

minutes that it plays. So we should be able to figure out how many revolutions it makes, and that will be the same as the number of "layers" on the record. For example, if a record plays for  $22\frac{1}{2}$  minutes at  $33\frac{1}{3}$  revolutions per minute, it will make 750 revolutions, and thus  $n$  will be 750. Furthermore, suppose that the inner circumference of the playing surface is 415 mm and the outer circumference is 917 mm. Then the points (1, 415) and (750, 917) will lie on the graph, and we can easily calculate that  $m = .067$  and  $b = 414.33$ . Each student should measure one of his own records and make the calculations.

The purpose of this approach was not merely to solve the problem, but to give the students a glimpse of what is involved in designing an experiment. They had to specify what they hoped to learn from the experiment, make appropriate measurements, organize the data, and discover a relationship. As in all experimental situations, it is necessary to deal with accuracy and acceptable error limits. This type of experience is most beneficial if it is done in groups so that students can make cooperative decisions.

As mentioned before, there are several theoretical approaches one can take to this problem. One rather naive approach is to deal with averages. Since we can easily find the circumferences of the inner and outer layers, we can find the average layer length. Then the total length of the groove is found as the product of the number of layers times the average layer length. This method gives a precise answer because, in this case, the average of all the layers is the average of the first and last layers. This is true whenever the increase in length

is linear, but does not have to be true in general.

Another method requires knowing the average groove thickness, a quantity which we have managed to avoid thus far. It is easily found, however, simply by dividing the radius of the playing surface by the total number of layers. If we let  $L$  = total groove length,  $t$  = average thickness of the groove,  $R_o$  = outer radius of the playing surface, and  $R_i$  = inner radius of the playing surface, then we can easily calculate the total grooved area as being  $\pi(R_o^2 - R_i^2)$ . Now imagine unwinding the groove so that it forms a long, thin rectangle  $L$  units long and  $t$  units wide. The area of this rectangle is  $Lt$ . Equating the two area expressions gives

$$L = \frac{\pi(R_o^2 - R_i^2)}{t}.$$

The solutions given so far have relied on the assumption that the groove of a record is composed of a series of concentric circles. This is a very good approximation, but the groove is really a continuous spiral. It would be more precise to describe the groove by an exponential spiral, whose equation is written in polar coordinates as  $r = ae^{b\theta}$ , where  $a$  and  $b$  are constants. Again, let  $R_o$  = outer radius,  $R_i$  = inner radius,  $L$  = total groove length, and  $n$  = number of revolutions the record makes. Imagine tracing the groove from the inside of the record to the outer edge. First we need to evaluate the constants  $a$  and  $b$ . We can do this by considering two points on the spiral. At the beginning of the spiral,  $\theta = 0$  and  $r = R_i$ . So  $R_i = ae^0 = a$ . At the end of the spiral,  $r = R_o$  and  $\theta = 2\pi n$ . In this case,  $R_o = R_i e^{(2\pi n)b}$ , which gives that

$$b = \frac{\ln R_o - \ln R_i}{2\pi n}.$$

Once we have evaluated  $a$  and  $b$ , we need to find the total length,  $L$ , of the spiral. From calculus, we know that if  $t$  is a parameter of a curve such that  $x = x(t)$  and  $y = y(t)$  where  $\alpha \leq t \leq \beta$ , then the length of the curve is given by  $L = \int_{\alpha}^{\beta} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right]^{\frac{1}{2}} dt$ . Using polar coordinates, we have  $x = r \cos \theta$  and  $y = r \sin \theta$ . For the exponential spiral,  $r = ae^{b\theta}$  so  $x = ae^{b\theta} \cos \theta$  and  $y = ae^{b\theta} \sin \theta$ . Since  $x = x(\theta)$  and  $y = y(\theta)$ , we can regard  $\theta$  as a parameter of the curve where  $0 \leq \theta \leq 2\pi n$ . Then

$$\begin{aligned} \frac{dx}{d\theta} &= abe^{b\theta} \cos \theta - ae^{b\theta} \sin \theta = ae^{b\theta} (b \cos \theta - \sin \theta) \\ \frac{dy}{d\theta} &= abe^{b\theta} \sin \theta + ae^{b\theta} \cos \theta = ae^{b\theta} (b \sin \theta + \cos \theta) \\ L &= \int_0^{2\pi n} \left[ (ae^{b\theta} (b \cos \theta - \sin \theta))^2 + (ae^{b\theta} (b \sin \theta + \cos \theta))^2 \right]^{\frac{1}{2}} d\theta \\ &= \int_0^{2\pi n} ae^{b\theta} (b^2 + 1)^{\frac{1}{2}} d\theta \\ &= \frac{a(b^2 + 1)^{\frac{1}{2}}}{b} (e^{2\pi nb} - 1) \end{aligned}$$

The theoretical solutions presented will depend on the level of the class. The measurements that the students made on the phonograph records should be used in the theoretical solutions to see if the results are consistent.

The experimental method with the roll of adding machine tape and the numerical measurements from such an experiment were given by James J. Hirstein, "Phonograph Records and Adding Machine Paper," Mathematics Teacher 73 (April 1980), p. 258 - 61. The suggestion to use the exponential spiral, although not the details of the solution, came from Joseph Budnitsky, "Reader Reflections," Mathematics Teacher 73 (October 1980), p. 489.

## POINSOT STARS

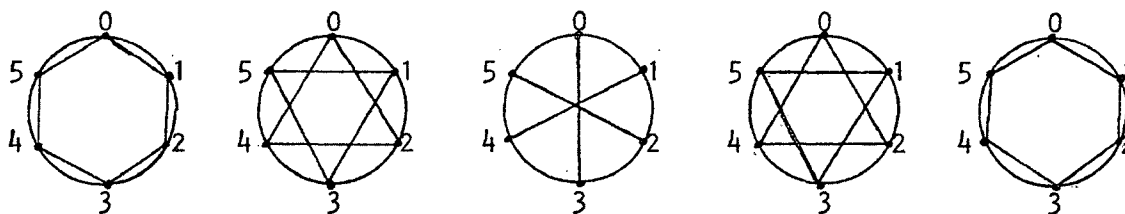
Pointsot stars are figures that are created by methodically connecting equally spaced points on a circle. They were first studied systematically by the French mathematician Louis Poinot (Pwañ.sō) in 1809.<sup>1</sup> Poinot stars offer a rich opportunity for mathematical exploration on many levels, from a simple geometrical pattern-finding activity to sophisticated investigations in number theory. How far an individual or a class proceeds into the topic will depend upon their mathematical background, abilities, and interests. At any level, familiarity with the properties of Poinot stars must come from seeing a great number of them actually drawn out. Usually this will be done by hand, and for that reason, a set of several circles is given which can be duplicated. If the class has access to a computer with graphics capabilities, it is possible to program the drawing.<sup>2</sup> Discovery of the properties of Poinot stars demands systematic record keeping, so the use of tables to arrange data is encouraged. This is a simple, though often overlooked, technique to aid in the synthesis of data.

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<sup>1</sup>The formal definition, historical background, and elementary properties of the stars are given in an article by Christian Hirsch, "Poinot Stars," Mathematics Teacher 73 (January 1980), p. 260-63.

<sup>2</sup>For a discussion of how this can be accomplished, see Harold Abelson and Andrea diSessa, Turtle Geometry (Cambridge: MIT Press, 1981), p. 3-50.

A set of Poinset stars is drawn as follows: Starting with  $n$  equally spaced points on a circle, connect each point to each adjacent point. On a second circle, connect each point to every second point. On the next circle connect each point to every third point, and so on. The set of Poinset stars with  $n = 6$  is shown.



The points on the circle have been numbered to help keep track of them. It is immaterial whether they are numbered  $0, 1, \dots, n - 1$  or  $1, 2, \dots, n$ , but in the discussion which follows, it is more convenient to start the numbering with 0. For the rest of the discussion, it is assumed that we always start drawing at point 0 and that we draw in a clockwise direction.

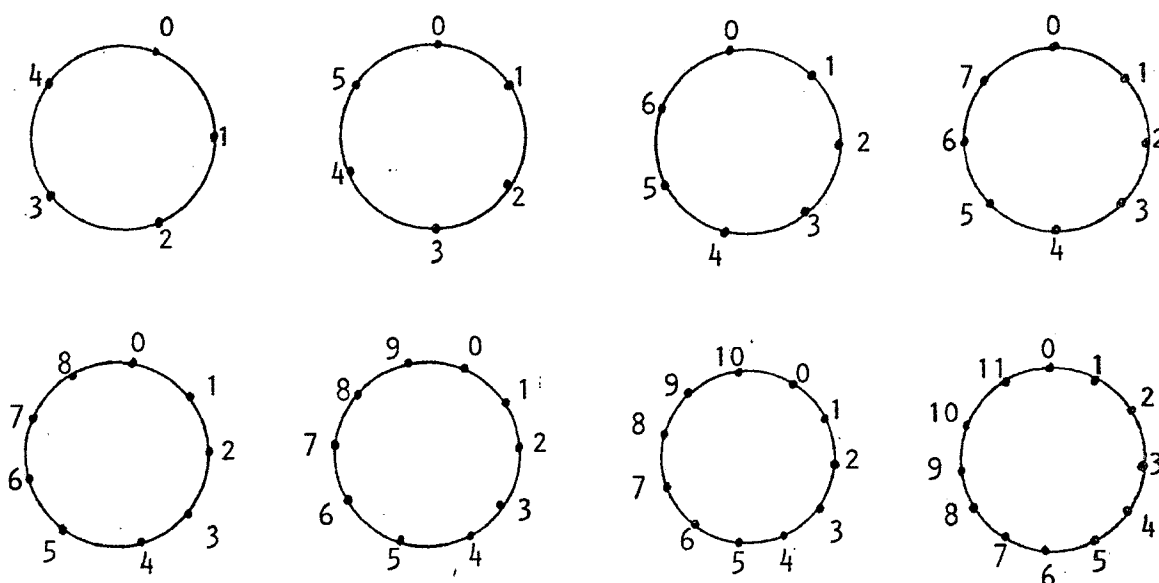
One immediate observation is that some of the stars look the same. By inspecting the examples given for  $n = 6$  above, it is apparent that the star obtained by connecting every 5th point is the same as that obtained by connecting every 1st point; it is just traced in the opposite direction. Also, the stars created by connecting every 2nd point and every 4th point end up looking the same. In general, the stars formed by connecting the  $d$ th points and the  $(n - d)$ th points are the same.

A natural question to ask is How many different stars are there for  $n$  points? One approach to use with lower level classes is to try and discover a rule by drawing a number of cases and constructing a table

such as this one:

Number of points	4	5	6	7	8	9	10	11	12
Number of stars	2	2	3	3	4	4	5	5	6

These patterns may be duplicated and given to the students for their drawings.



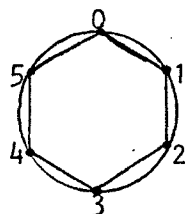
Another approach for more advanced classes is to reason deductively from the fact that stars formed by connecting the  $d$ th points and the  $(n - d)$ th points look the same. Point 0 can be connected to  $n - 1$  other points, so there are  $n - 1$  possible stars. If  $n$  is odd, there are an even number of stars and each one occurs twice, so there are  $\frac{n - 1}{2}$  different stars. If  $n$  is even, there are an odd number of stars, so they cannot all occur twice. In this case, connecting point 0 to point  $\frac{n}{2}$  is exactly the same as connecting point 0 to point  $(n - \frac{n}{2})$  so that particular star only occurs once. Half of the remaining  $n - 2$  connec-



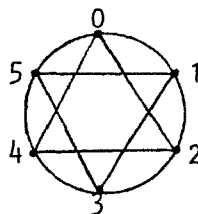
tions will give distinct stars, so there are  $\frac{n-2}{2} + 1 = \frac{n}{2}$  different stars.

Another conclusion can be drawn from the fact that the same star is formed by connecting the  $d$ th points and the  $(n-d)$ th points. Notice that  $d + (n-d) = n$ . If  $n = 8$ , for example, we could write  $1 + 7 = 8$ ,  $2 + 6 = 8$ ,  $3 + 5 = 8$ , and  $4 + 4 = 8$ . We could interpret this by saying that the star constructed by connecting every 1st point is the same as that constructed by connecting every 7th point; the one obtained by connecting every 2nd point is the same as the one obtained by connecting every 6th point, and so on. Each pair of positive integers that adds up to  $n$  represents a different star. So the number of Poinot stars determined by  $n$  points is the number of distinct ways  $n$  can be represented as the sum of two positive integers. Therefore, if  $n$  is even we know that there are  $\frac{n}{2}$  distinct pairs of positive integers that add up to  $n$ , and if  $n$  is odd there are  $\frac{n-1}{2}$  such pairs.

An  $n$ -point Poinot star is called regular if all points are reached before returning to point 0. For example, a 6-point star where  $d = 1$  is regular. A 6-point star with  $d = 2$  is a non-regular because only 3 of the points can be connected before we return to 0.



regular



non-regular

The student could make a table such as the following to discover what conditions will produce a regular star.

n	d	regular?
4	1 or 3	yes
	2	no
5	1 or 4	yes
	2 or 3	yes
6	1 or 5	yes
	2 or 4	no
	3	no
7	1 or 6	yes
	2 or 5	yes
	3 or 4	yes
8	1 or 7	yes
	2 or 6	no
	3 or 5	yes
	4	no
9	1 or 8	yes
	2 or 7	yes
	3 or 6	no
	4 or 5	yes

By drawing stars and filling out such a table, the student should hypothesize that  $n$  and  $d$  must be relatively prime to produce a regular star. He might also realize that if vertex 1 is reached without lifting the pencil, then the star will be regular.

Given our starting assumptions, a proof of why  $n$  and  $d$  must be relatively prime requires the use of residue classes modulo  $n$ . It is possible to prove the result if one changes perspectives and considers the exterior angles of the polygons formed and how many revolutions of the circle are needed to complete the figure. Using this analysis, it is possible to derive the important number theory result that two integers  $n$  and  $d$  are relatively prime if and only if there exist integers  $p$  and  $q$  such that  $pn + qd = 1$ . For such a treatment, see Abelson and diSessa (1981).

How many different regular stars are there? We get a regular star each time  $n$  and  $d$  are relatively prime, so each number less than and

relatively prime to  $n$  produces a regular star. If we get a regular star for  $d$ , we get the same star for  $n - d$ . So the number of distinct regular stars is half the number of integers less than and relatively prime to  $n$ . If  $n = 7$ , for example, there are six numbers (1, 2, 3, 4, 5, and 6) less than and relatively prime to 7, so there must be three distinct regular stars. There are four numbers less than and relatively prime to 8 (1, 3, 5, and 7), so there must be two regular 8-point stars.

A formula for how many numbers are less than and relatively prime to  $n$  was first proved by Euler, and is called Euler's  $\phi$ -function. Let us first consider the simplest possible situation:  $n$  is prime. Then all the  $n - 1$  numbers  $\{1, 2, \dots, n - 1\}$  are less than and relatively prime to  $n$ , so  $\phi(n) = n - 1$ . Now suppose that  $n$  is a power of a prime,  $n = p^\alpha$ . Then there are  $p^\alpha - 1$  numbers less than  $n$ . The only ones not relatively prime to  $n$  are the multiples of  $p$ ,  $\{p, 2p, 3p, \dots, (p^{\alpha-1} - 1)p\}$ , and there are  $p^{\alpha-1} - 1$  of these. So  $\phi(n) = p^\alpha - 1 - (p^{\alpha-1} - 1) = p^\alpha - p^{\alpha-1} = p(1 - \frac{1}{p})$ . The final step is to consider what happens when  $n$  is composite, i.e. it can be factored into primes as  $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ . Generalizing  $\phi(n)$  for composite cases requires that we show  $\phi$  is multiplicative, i.e. if  $a$  and  $b$  are relatively prime then  $\phi(ab) = \phi(a)\phi(b)$ . A relatively uncomplicated proof that  $\phi$  is multiplicative can be found in Underwood Dudley, Elementary Number Theory (San Francisco: W. H. Freeman & Company, 1969), p. 66-68. Even this proof requires more than high school mathematics, and the result will be taken on faith here. Once we know that  $\phi$  is multiplicative, a general formula for  $\phi(n)$  is easy to derive from what we already know about  $\phi(p^\alpha)$ . We have

$$\begin{aligned}
 \phi(n) &= \phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}) \\
 &= \phi(p_1^{\alpha_1}) \phi(p_2^{\alpha_2}) \dots \phi(p_k^{\alpha_k}) \\
 &= p_1^{\alpha_1} \left(1 - \frac{1}{p_1}\right) p_2^{\alpha_2} \left(1 - \frac{1}{p_2}\right) \dots p_k^{\alpha_k} \left(1 - \frac{1}{p_k}\right) \\
 &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right)
 \end{aligned}$$

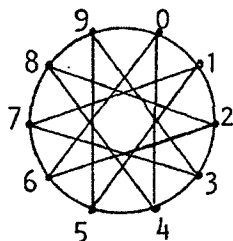
The non-regular Poinset stars offer some interesting areas of investigation. Obviously, for non-regular stars  $n$  and  $d$  will have a greatest common divisor greater than 1. A non-regular star is formed when we return to 0 before we have connected all the points, therefore it will be composed of two or more superimposed regular stars. Are the greatest common divisor of  $n$  and  $d$ , the number of superimposed regular stars, and the number of sides of each regular star related? A starting point might be to organize some observational data in a table similar to the following:

$n$	$d$	$\gcd(n,d)$	number of regular stars ( $r$ )	number of sides of each regular star ( $s$ )
10	2 or 8	2	2	5
	4 or 6	2	2	5
	5	5	5	*
12	2 or 10	2	2	6
	3 or 9	3	3	4
	4 or 8	4	4	3
	6	6	6	*

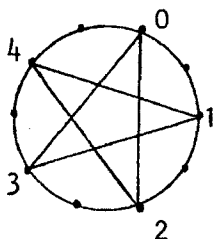
\* These stars are lines so we cannot give a number of sides.

Some possible relationships are immediately suggested by this table. It should be fairly apparent why the number of regular stars,  $r$ , times the number of sides of each star,  $s$ , has to be  $n$ . Each  $s$ -sided regular star touches  $s$  points on the circle, and there are  $r$  superimposed regular stars making up the non-regular star. All  $n$  points on the circle must be touched, so we have that  $rs = n$ .

To understand the role of the greatest common divisor, consider the non-regular star formed when  $n = 10$  and  $d = 4$ .

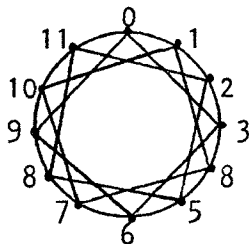


It is made up of two regular stars using  $n = 5$  and  $d = 2$ .

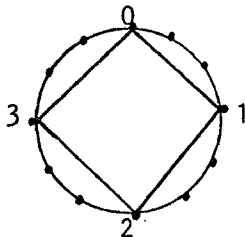


If we start with a 5-point circle and add 5 more points to make a 10-point circle, we double  $n$  and we also double  $d$ . Now there is room to superimpose the second 5-sided star to make the  $n = 10$ ,  $d = 4$  star.

Now consider the non-regular star formed for  $n = 12$ ,  $d = 3$ .



It is made of 3 superimposed stars for which  $n = 4$ ,  $d = 1$ .



If we multiply  $n$  by 3, we also multiply  $d$  by 3, because point 1 will become point 3.

Now consider a non-regular star made up of  $r$  superimposed regular stars. Each regular star can be drawn on a circle using  $n'$  equally spaced points and connecting every  $d'$ th one. Since the star is regular,  $n'$  and  $d'$  will be relatively prime. Now multiply  $n'$  by  $r$ , which also has the effect of multiplying  $d'$  by  $r$ . There are now enough points on the circle to superimpose  $r$  regular stars and form a non-regular star for which  $n = rn'$  and  $d = rd'$ . Since  $n'$  and  $d'$  are relatively prime, the greatest common divisor of  $n$  and  $d$  must be  $r$ .

Some students might be interested in creating artistic designs by superimposing various combinations of Poinot stars. For example, if all the Poinot stars determined by 24 points are drawn on a single circle, an illusion of a series of concentric circles is created. For further exploration in this direction, see Phil Locke, "Residue Designs," Mathematics Teacher 65 (March 1972), p. 260-63.

The properties of Poinot stars offer a rich area of exploration. The student can carry the investigation as far as his interest allows, and it can lead to quite sophisticated mathematics. The topic also affords a prime area for exploring mathematical concepts through the use of computer graphics.

## FAREY SEQUENCES

High school students usually do not have the vaguest idea of how mathematics is really developed. The properties and proofs in the textbook are neatly presented without any indication of how they have been discovered. The problems that the teacher does for illustration are carefully selected to illustrate a particular principle and work out very nicely. This gives the students the impression that mathematicians always know the proper approach to a problem, can immediately write a concise and elegant proof, and recognize general principles without effort. No wonder many of them view mathematicians as a breed apart!

This exercise with Farey sequences is designed to give students some experience in how mathematics is actually done. Students should not attempt this problem without some previous problem solving experience. They should feel comfortable with open-ended investigations, with generalizing results, and be experienced in searching for patterns.

A Farey sequence,  $F_n$ , is a sequence of fractions beginning with  $\frac{0}{1}$ , ending with  $\frac{1}{1}$  and including in ascending order all the proper irreducible fractions with denominators  $m$  such that  $2 \leq m \leq n$ . The first four Farey sequences are given below.

$$F_1 \text{ is } \frac{0}{1}, \frac{1}{1}.$$

$$F_2 \text{ is } \frac{0}{1}, \frac{1}{2}, \frac{1}{1}.$$

$$F_3 \text{ is } \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}.$$

$$F_4 \text{ is } \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}.$$

Give the students these four sequences to illustrate the definition. Their assignment is then to write  $F_5$  and to discover one or more general properties of Farey sequences. The next day in class make a list of their discoveries. Some of them will need help stating their properties in a precise and clear language, and some properties are sure to appear in more than one form. Better students might be interested in formally proving some of the properties; others might want to write out some additional sequences to see if the patterns still hold. It is impossible to anticipate exactly which properties will be discovered, but some common ones are discussed below.

Property 1: If  $\frac{a}{b}$  and  $\frac{c}{d}$  are two consecutive fractions of  $F_n$ , then  $bc - ad = 1$ . This is a property that should be pointed out to students if they do not discover it themselves. Proving this property is fairly difficult, but once the property is assumed, it can be used to prove some of the subsequent properties. Two number theoretical proofs can be found in G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers (London: Oxford University Press, 1945), p. 24. A proof using two-dimensional lattices which could be adapted to a geoboard is given in H. S. M. Coxeter, Introduction to Geometry, 2nd ed. (New York: John Wiley & Sons, 1969), P. 210.

Property 2: If  $\frac{a}{b}$ ,  $\frac{c}{d}$ ,  $\frac{e}{f}$  are three consecutive fractions, then  $\frac{c}{d} = \frac{a + e}{b + f}$ . Proving this is the same thing as showing that  $c(b + f) = d(a + e)$ , or equivalently, that  $bc + cf = ad + de$ . The only tools we have to use for a proof are the definition and property 1. The definition does not tell us anything about products such as  $bc$  and  $cf$ , but property 1



does. Applying property 1 to the three consecutive fractions, we see that  $bc - ad = 1$  and  $de - cf = 1$ . Our final equation does not contain a 1 so we want to eliminate that. We can do so by using the transitive property to get  $bc - ad = de - cf$ . This looks very close to the equation we wanted ( $bc + cf = ad + de$ ) and, in fact, is an equivalent equation. We can now put together the following formal proof of property 2:

$$bc - ad = 1 \text{ and } de - cf = 1$$

$$bc - ad = de - cf$$

$$bc + cf = ad + de$$

$$c(b + f) = d(a + e)$$

$$\frac{c}{d} = \frac{a + e}{b + f}$$

Property 2 can be generalized as follows: If  $\frac{a}{b}$  and  $\frac{c}{d}$  are any two non-consecutive terms of  $F_n$ , and if  $b + d \leq n$ , then  $\frac{a + c}{b + d}$  is also a term of  $F_n$  and is between  $\frac{a}{b}$  and  $\frac{c}{d}$ . By the definition of the sequence, if  $b + d \leq n$  then  $\frac{a + c}{b + d}$  would be a term of  $F_n$ . We can discover a proof of this generalization by working backwards, as we did for the proof of property 2. Showing that  $\frac{a}{b} < \frac{a + c}{b + d}$  is equivalent to showing that  $ab + ad < ab + bc$ . Since  $ab$  appears on both sides of this inequality, we only need show that  $ad < bc$ . Making fractions in this inequality, we get that  $\frac{a}{b} < \frac{c}{d}$ . We may assume that  $\frac{a}{b}$  is the smaller fraction, so a proof looks like this:

$$\frac{a}{b} < \frac{c}{d}$$

$$ad < bc$$

$$ad + ab < bc + ab$$

$$a(b + d) < b(a + c)$$

$$\frac{a}{b} < \frac{a + c}{b + d}$$

Similarly, we can show that  $\frac{a + c}{b + d} < \frac{c}{d}$  so  $\frac{a + c}{b + d}$  always has to fall between  $\frac{a}{b}$  and  $\frac{c}{d}$ . Notice that this proof used only the order of the fractions and not any of the properties of Farey sequences. Thus, the fraction  $\frac{a + c}{b + d}$  will always fall between the fractions  $\frac{a}{b}$  and  $\frac{c}{d}$  whether or not these fractions are members of a Farey sequence. These three fractions will be members of all sequences  $F_n$  where  $n \geq b + d$ .

It is also possible to show that of all rational numbers between  $\frac{a}{b}$  and  $\frac{c}{d}$ ,  $\frac{a + c}{b + d}$  is the one with the smallest denominator. In other words, if  $\frac{a}{b}$  and  $\frac{c}{d}$  are consecutive terms in a Farey sequence,  $\frac{a + c}{b + d}$  will be the first term inserted between them in a subsequent sequence and will appear in  $F_{b + d}$ . To prove this, suppose that  $\frac{x}{y}$  is any other fraction such that  $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$ . We need to show that  $y > b + d$ . The difficult part of this proof is discovering how to get started. We need some sort of relationship involving  $y$ ,  $b$  and  $d$ , and it will probably have to involve  $a$ ,  $x$ , and  $c$  also. Using what we have developed so far, the most explicit relationships come from property 1. However, property 1 depends on the terms being consecutive, an assumption we cannot make about  $\frac{a}{b}$ ,  $\frac{x}{y}$ , and  $\frac{c}{d}$ . The only information we have is that  $\frac{a}{b} < \frac{x}{y}$  and  $\frac{x}{y} < \frac{c}{d}$ . From the first inequality we can get that  $ay < bx$  or that  $bx - ay > 0$ . Now  $bx - ay$  has to be an integer, so we can rephrase this as  $bx - ay \geq 1$ . In the same manner, we can get that  $cy - dx \geq 1$ . This looks similar to the relationship in property 1, only it involves an inequality, which we want. Now, how can we use this relationship? We need to subtract things, and we

have to get terms like  $bx$  and  $ay$ . We could do this by subtracting fractions and making common denominators. Some experimentation might be needed at this point to discover the best way to proceed. One way of involving all three fractions and of subtracting is to write  $\frac{c}{d} - \frac{a}{b} = (\frac{c}{d} - \frac{x}{y}) + (\frac{x}{y} - \frac{a}{b}) = \frac{cy - dx}{dy} + \frac{bx - ay}{by}$ . Now we can use the relationships we just discovered,  $cy - dx \geq 1$  and  $bx - ay \geq 1$ , to write  $\frac{c}{d} - \frac{a}{b} \geq \frac{1}{dy} + \frac{1}{by}$ , which is equivalent to  $\frac{c}{d} - \frac{a}{b} \geq \frac{b+d}{bdy}$ . The right-hand side of this last inequality has the quantity  $b+d$  on the small side, which is one of the things we want. (Remember that we wish to prove  $y > b+d$ .) So let us see what can be done with the left-hand side. We can write  $\frac{c}{d} - \frac{a}{b}$  as  $\frac{cb - ad}{bd}$ . Now there is some Farey sequence where  $\frac{a}{b}$  and  $\frac{c}{d}$  appear as consecutive terms. Therefore,  $cb - ad = 1$ , and we can write  $\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}$ . Now we have  $\frac{1}{bd} \geq \frac{b+d}{bdy}$ . We need a  $y$  on the left-hand side, so multiply by  $\frac{y}{y}$  to get  $\frac{y}{bdy} \geq \frac{b+d}{bdy}$ , and from this we can get that  $y \geq b+d$ . If  $y = b+d$  then  $x = z+c$  (see property 3 below).

Property 3: No two consecutive terms of  $F_n$  have the same denominator. Suppose otherwise, i.e. that  $\frac{a}{c}$  and  $\frac{b}{c}$  are two consecutive terms of  $F_n$  with  $a < b$ . Then by property 1,  $bc - ac = 1$  so  $c = \frac{1}{b-a}$ . Now  $c$  has to be an integer, so  $b-a = 1$ . This means that  $c = \frac{1}{b-a} = 1$ , and the only term with denominator of 1 is  $\frac{1}{1}$ . Therefore,  $\frac{a}{c} = \frac{b}{c} = 1$ , a contradiction.

Property 4: The sum of two consecutive denominators in  $F_n$  is always greater than  $n$ . This statement is a direct result of property 2. If  $\frac{a}{b}$  and  $\frac{c}{d}$  are two terms of  $F_n$ , then we know that  $\frac{a+c}{b+d}$  is between them. However, if  $\frac{a}{b}$  and  $\frac{c}{d}$  are consecutive terms of  $F$  then  $\frac{a+c}{b+d}$  is not a term

of  $F_n$ . The only thing that would keep  $\frac{a+c}{b+d}$  from being a term of  $F_n$  is if  $b+d > n$ .

Property 5: A useful characteristic for doing some of the problems which follow is to note that Farey sequences form nested sets, i.e.

$F_1 \subset F_2 \subset \dots \subset F_n \subset F_{n+1} \subset \dots$ . This is a direct consequence of the definition.

Property 6: For  $n > 1$ , the term  $\frac{1}{2}$  always appears in the middle of the sequence. Terms that are in symmetric positions with respect to  $\frac{1}{2}$  always add up to one. More precisely, the  $r$ th term and the  $(n - r + 1)$ th term always add up to one. This observation will shorten the work involved in generating a Farey sequence.

An induction argument can be used to justify property 6. Certainly the property is true for  $F_2$ . Suppose it is true for  $F_{b+d-1}$ . To form  $F_{b+d}$  from  $F_{b+d-1}$  we must insert all fractions with denominator  $b+d$ .  $F_{b+d-1}$  will look like

$$\frac{0}{1} \dots \frac{a}{b} \frac{c}{d} \dots \frac{1}{2} \dots \frac{b-a}{b} \frac{d-c}{d} \dots \frac{1}{1}.$$

A fraction with denominator  $b+d$  can be inserted between  $\frac{a}{b}$  and  $\frac{c}{d}$  and also between  $\frac{b-a}{b}$  and  $\frac{d-c}{d}$ . (Note that there may be fractions other than these two inserted, as there could be other pairs of denominators that add up to  $b+d$ . For instance, in  $F_5$   $\frac{1}{5}$  is inserted between  $\frac{0}{1}$  and  $\frac{1}{4}$ , and  $\frac{2}{5}$  is inserted between  $\frac{1}{3}$  and  $\frac{1}{2}$ . The same argument applies to any other inserted fractions.) These inserted fractions are  $\frac{a+c}{b+d}$  and  $\frac{b-a+d-c}{b+d}$ . They are symmetrically placed with respect to  $\frac{1}{2}$  and  $\frac{a+c}{b+d} + \frac{b-a+d-c}{b+d} = 1$ . Therefore, the property holds for  $F_{b+d}$  and by induction holds for all  $F_n$ .

The following page contains some exercises which give the student an opportunity to apply some of these properties. These exercises and more can be found in George S. Cunningham, "Farey Sequences" in Enrichment Mathematics for High School, Twenty-eight Yearbook of the National Council of Teachers of Mathematics (Washington: National Council of Teachers of Mathematics, 1963), p. 2. Additional exercises can be found in Ivan Niven and Herbert S. Zuckerman, An Introduction to the Theory of Numbers, 3rd ed. (New York: John Wiley & Sons, 1972), p. 137. Students who are interested in investigating this topic further might wish to explore the relationship between Farey sequences and Euler's  $\phi$ -function. Farey sequences also appear as approximations to irrational numbers in the study of continued fractions. For a treatment of this topic see C. D. Olds, Continued Fractions (New York: Random House, 1963).

## EXERCISES FOR FAREY SEQUENCES

1. For a certain  $n$ ,  $F_n$  has exactly one term between  $\frac{2}{3}$  and  $\frac{3}{4}$ . Find the missing term, and decide what  $n$  is. Is more than one sequence possible? How many?
2. For a certain  $n$ ,  $F_n$  has exactly two terms between  $\frac{2}{3}$  and  $\frac{3}{4}$ . Find the missing terms and find  $n$ . Is more than one sequence possible? How many?
3. Generate  $F_6$  by using the first term  $\frac{0}{1}$  and the last term  $\frac{1}{1}$  and applying properties 2 and 6.
4. If  $n > 1$ , will  $F_n$  have an odd or even number of terms?

## SOLUTIONS

1. The term between  $\frac{2}{3}$  and  $\frac{3}{4}$  has to be  $\frac{2+3}{3+4} = \frac{5}{7}$ .  $\frac{5}{7}$  appears in all  $F_n$  where  $n \geq 7$ . If there was one term between  $\frac{2}{3}$  and  $\frac{5}{7}$  it would be  $\frac{7}{10}$  and the first term between  $\frac{5}{7}$  and  $\frac{3}{4}$  is  $\frac{8}{11}$ . So  $\frac{5}{7}$  is the only term between  $\frac{2}{3}$  and  $\frac{3}{4}$  in  $F_7$ ,  $F_8$ , and  $F_9$ .
2. Using the same analysis as in problem 1, we see that in  $F_{10}$  the terms  $\frac{7}{10}$  and  $\frac{5}{7}$  appear between  $\frac{2}{3}$  and  $\frac{3}{4}$ . In  $F_{11}$  there are three terms between  $\frac{2}{3}$  and  $\frac{3}{4}$  so  $F_{10}$  is the only sequence that will satisfy the conditions of the problem.
3. We use property 2 repeatedly to generate the first half of the sequence, then use property 6 to construct the second half.

$$\begin{array}{ccccccc}
 0 & & & & & & 1 \\
 \frac{0}{1} & & & & & & \frac{1}{2} \\
 0 & & & 1 & & & 1 \\
 \frac{0}{1} & & & \frac{1}{3} & & & \frac{1}{2} \\
 0 & & 1 & 1 & & & 1 \\
 \frac{0}{1} & & \frac{1}{4} & \frac{1}{3} & & & \frac{1}{2} \\
 0 & 1 & 1 & 1 & 2 & 1 & \\
 \frac{0}{1} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \\
 0 & 1 & 1 & 1 & 1 & 2 & 1 \\
 \frac{0}{1} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} \\
 0 & 1 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 5 & 1 \\
 \frac{0}{1} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \frac{1}{1}
 \end{array}$$

4. Property 6 tells us that for  $n > 1$ ,  $F_n$  must always have an odd number of terms.

## CYCLIC FRACTIONS -- EMPIRICAL EXPLORATIONS

High school students generally do not appreciate the potential of arithmetic as an area for mathematical investigation. They regard arithmetic as a set of computational rules and rarely take an opportunity to explore and generalize numerical relationships. This chapter and the following one describe an investigation which gives students a chance to become acquainted with some of the number theory behind the decimal expansions of rational numbers. This investigation can be carried out on many levels, depending on the interest and abilities of the students. This chapter describes an empirical approach to the problem, which requires the students to organize data, discover patterns, generalize results, and make conjectures. The following article, "Cyclic Fractions -- Theoretical Considerations," explains the empirical results in terms of congruences. To understand the theoretical solutions, students must have some knowledge of modular systems. All students, however, will benefit from the numerical explorations.

This problem provides a good opportunity for students to carry out their own investigations, with advice and help from the teacher as needed. There are many different avenues of exploration and several ways of attacking the problem. Students should not attempt such a project without some previous problem solving experience. They should be familiar with different problem solving strategies and should have seen examples of how to generalize a result. It is suggested that the students work



together as a class or in groups in order to keep numerical tedium from overwhelming mathematical curiosity.

The study begins with an observation of the following coincidence.

$$\frac{1}{7} = .\overline{142857}$$

$$\frac{3}{7} = .\overline{428571}$$

The similarity between the two decimal expansions is apparent. They contain the same digits in the same cyclic order. The mathematician does not believe in mere coincidence and immediately becomes curious about this hint of a pattern and what causes it. A sensible place to begin the investigation is to write out the decimal expansions for all the proper fractions with 7 as denominator. The results are given below.

$$\frac{1}{7} = .\overline{142857}$$

$$\frac{2}{7} = .\overline{285714}$$

$$\frac{3}{7} = .\overline{428571}$$

$$\frac{4}{7} = .\overline{571428}$$

$$\frac{5}{7} = .\overline{714285}$$

$$\frac{6}{7} = .\overline{857142}$$

Our suspicion of a pattern has been confirmed! All the decimal expansions consist of six repeating digits in the same cyclic order. We shall call fractions whose decimal expansions have this property cyclic fractions, and we shall refer to the set  $\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}$  as a fraction set. The obvious question to be asked is, Are there any other sets of cyclic fractions? For the rest of the discussion, let  $n$  be the denomina-

tor of a fraction and assume that we are only considering proper fractions.

A necessary condition for a fraction to be cyclic can be determined immediately. There are  $(n - 1)$  fractions in the fraction set with denominator  $n$ . If this is to be a set of cyclic fractions, each one must start at a different point in the cycle, so there must be  $(n - 1)$  digits in the cycle.

Is this condition also sufficient? Yes, it is, and to see why, the students should do some of the divisions by hand. Have them pay particular attention to the remainders after each step in the long division process. The remainders are circled in the examples below.

$$\begin{array}{r}
 .142857 \\
 7 \overline{) 1.000000} \\
 \underline{7} \phantom{000000} \\
 \textcircled{3} 0 \phantom{00000} \\
 \underline{28} \phantom{00000} \\
 \textcircled{2} 0 \phantom{00000} \\
 \underline{14} \phantom{00000} \\
 \textcircled{6} 0 \phantom{00000} \\
 \underline{56} \phantom{00000} \\
 \textcircled{4} 0 \phantom{00000} \\
 \underline{35} \phantom{00000} \\
 \textcircled{5} 0 \phantom{00000} \\
 \underline{49} \phantom{00000} \\
 \textcircled{1}
 \end{array}$$

$$\begin{array}{r}
 .285714 \\
 7 \overline{) 2.000000} \\
 \underline{14} \phantom{000000} \\
 \textcircled{6} 0 \phantom{00000} \\
 \underline{56} \phantom{00000} \\
 \textcircled{4} 0 \phantom{00000} \\
 \underline{35} \phantom{00000} \\
 \textcircled{5} 0 \phantom{00000} \\
 \underline{49} \phantom{00000} \\
 \textcircled{1} 0 \phantom{00000} \\
 \underline{7} \phantom{00000} \\
 \textcircled{3} 0 \phantom{00000} \\
 \underline{28} \phantom{00000} \\
 \textcircled{2}
 \end{array}$$

$$\begin{array}{r}
 .428571 \\
 7 \overline{) 3.000000} \\
 \underline{28} \phantom{000000} \\
 \textcircled{2} 0 \phantom{00000} \\
 \underline{14} \phantom{00000} \\
 \textcircled{6} 0 \phantom{00000} \\
 \underline{56} \phantom{00000} \\
 \textcircled{4} 0 \phantom{00000} \\
 \underline{35} \phantom{00000} \\
 \textcircled{5} 0 \phantom{00000} \\
 \underline{49} \phantom{00000} \\
 \textcircled{1} 0 \phantom{00000} \\
 \underline{7} \phantom{00000} \\
 \textcircled{3}
 \end{array}$$

If we are dividing by  $n$ , there are only  $(n - 1)$  possible remainders: 1, 2, ...,  $n - 1$ . The expansion will start to repeat when a remainder appears for the second time. So if the expansion has  $(n - 1)$  digits, we must have used all  $(n - 1)$  possible remainders. Once we get any remainder, the other remainders, and hence the other digits in the expansion, will follow in the same sequence. In dividing  $\frac{1}{n}$ ,  $\frac{2}{n}$ , ...,

$\frac{n-1}{n}$ , we essentially start with the remainders 1, 2, ...,  $n-1$ . So each expansion will start with a different digit, and the others will then follow in the same order. Thus, if we are only searching for cyclic fractions, it is enough to determine if the expansion of  $\frac{1}{n}$  has  $(n-1)$  digits.

Have the class find decimal expansions of  $\frac{1}{n}$  for  $2 \leq n \leq 20$  determine candidates for cyclic fractions. Let them split up the work so that it does not become too tedious; the point of this exercise is to encourage mathematical exploration, not to practice long division. They should find that  $n = 17$  and  $n = 19$  also give cyclic fraction sets.

This result does not give much information about which numbers  $n$  will generate sets of cyclic fractions. Furthermore, the search is getting very cumbersome and tedious. Perhaps it would pay to take a closer look at the expansions of  $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$  for all  $2 \leq n \leq 20$ . Often we can gain a great deal of insight by observing why numbers fail to satisfy a given criterion; in this case, What prevents them from being cyclic fractions? Again, the students should divide up the work. When all expansions have been completed, the work should be assembled in some convenient form so that each student can study a complete set of data.

When searching for patterns, it is helpful to be able to group the data into categories. When  $n = 2, 4, 5, 8, 10, 16$ , or  $20$ , all the decimals terminate. For  $n = 3, 7, 9, 11, 13, 17$ , or  $19$ , all the decimals repeat. When  $n = 6, 12, 14, 15$ , or  $18$ , the decimals for a single  $n$  were of mixed types: some terminated, some repeated, and some were delayed

repeaters (decimals that do not start their repeating pattern right away). The results are summarized below.

<u>n</u>	<u>type of expansions</u>	<u>n</u>	<u>type of expansions</u>
2	terminating	12	mixed
3	repeating	13	repeating
4	terminating	14	mixed
5	terminating	15	mixed
6	mixed	16	terminating
7	repeating (cyclic)	17	repeating (cyclic)
8	terminating	18	mixed
9	repeating	19	repeating (cyclic)
10	terminating	20	terminating
11	repeating		

Some patterns begin to emerge, although it is the exceptions to the patterns that provide the most insight. All the prime numbers have repeating patterns except for  $n = 2$  and  $n = 5$ . It should be obvious to students that these decimals terminate because 2 and 5 are factors of 10. Furthermore, they are the only prime factors of 10 so they are the only prime numbers that have terminating decimal expansions.

What about the other numbers with terminating expansions? When  $n = 4, 8, 10, 16$ , or 20, the expansions also terminate. Students should quickly realize that 4, 8, and 16 are powers of 2, and 10 and 20 are multiples of 10. A preliminary conjecture might be that all multiples of 10 and powers of 2 will have terminating expansions. A counter example can be quickly found for the multiples of 10, since  $\frac{1}{30}$  is a delayed repeater. However, all powers of 2 appear to be terminating, so we will keep that part of the conjecture. What about powers of 5? The second power of 5 was not listed, but it too will have a terminating expansion, as will  $5^3$ . So a revised conjecture could be that all powers

of 2 and all powers of 5 terminate. How could this conjecture be expanded to account for 10 and 20? Since  $10 = 2 \cdot 5$  and  $20 = 2^2 \cdot 5$ , maybe products of powers of 2 and powers of 5 terminate. We can formulate the conjecture as: When  $n = 2^r 5^s$  for some  $r$  and  $s$ , all fractions with denominator  $n$  will have terminating expansions.

To prove that all fractions with denominator  $n = 2^r 5^s$  terminate, it is really only necessary to prove that  $\frac{1}{n}$  terminates. Any other fraction  $\frac{a}{n} = a(\frac{1}{n})$ , so if  $\frac{1}{n}$  terminates  $a(\frac{1}{n})$  will terminate also. The proof is easier to understand if we consider some simpler cases. First consider  $\frac{1}{2^r}$ . We can write this as a division problem,  $2^r \overline{)1.0000\dots}$ , and ignore the decimal point for a moment. If this division is going to terminate, then we must have that  $10^y \div 2^r$  is an integer for some power  $y$ . But  $10^y = 2^y 5^y$ , and if we let  $y = r$ , then  $2^r 5^r \div 2^r$  is an integer. The same argument holds for  $\frac{1}{5^s}$ .  $10^y \div 5^s$  has to be an integer for some  $y$ , and it will be if  $y = s$ . Now what about  $\frac{1}{2^r 5^s}$ ? ( $10^y \div 2^r 5^s = 2^y 5^y \div 2^r 5^s$  will be an integer if  $y$  is the maximum of  $r$  and  $s$ . Students might need to do a few specific examples to become convinced of this argument.

An alternate approach showing that  $\frac{1}{2^r 5^s}$  has a terminating expansion is as follows. Any terminating decimal can be written as a fraction whose denominator is a power of 10. Therefore, a fraction with a terminating expansion must be equivalent to one whose denominator is a power of 10. We can make  $2^r 5^s$  into a power of 10 in the following manner:

$$\begin{aligned} \text{If } r \geq s, \quad \frac{1}{2^r 5^s} &= \frac{5^{r-s}}{2^r 5^s 5^{r-s}} = \frac{5^{r-s}}{2^r 5^r} = \frac{5^{r-s}}{10^r}. \\ \text{If } r < s, \quad \frac{1}{2^r 5^s} &= \frac{2^{s-r}}{2^r 5^s 2^{s-r}} = \frac{2^{s-r}}{2^s 5^s} = \frac{2^{s-r}}{10^s}. \end{aligned}$$

This proof gives perhaps a clearer explanation of why the final power of 10 is the maximum of  $r$  and  $s$ . In order to make it more plausible, it is advisable to consider first some numerical values of  $r$  and  $s$ , and then generalize the procedure to the two cases  $r \geq s$  and  $r < s$ .

We have now arrived at some conclusions about the terminating and the repeating fractions. What about those  $n$  which give a mixture of decimal expansions? All the mixed expansions occur when  $n$  is a composite number. (A unique case occurs for  $n = 9$ ; the number is composite but all the expansions are repeating. We shall set this case aside and consider it in a moment.) If  $n$  is composite, then some of the fractions in the fraction set will reduce. These reduced fractions will have the characteristic expansion of their reduced denominator. For instance, if  $n = 12$ , then those fractions which reduce to denominators of 2 or 4 will terminate, those which reduce to denominators of 3 or 6 will repeat, and those which do not reduce will be delayed repeaters.

Now what about  $n = 9$ ? If the fraction does not reduce, it repeats. If the fraction does reduce, the only thing it can reduce to has a denominator of 3, and those fractions also repeat. So the reason that all decimal expansions repeat when  $n = 9$ , even though 9 is composite, is the fact that the only factor of 9 is 3.

We seem to have reached the general conclusion that when  $n$  is a prime other than 2 or 5, the fraction  $\frac{m}{n}$  will have a repeating decimal expansion. Some of these will be cyclic fractions, but not all of them. We have narrowed the field in our search for cyclic fractions, but we have not made much headway in determining exactly when they occur.

There are two ways of continuing the investigation. We could further study the non-cyclic primes between 2 and 20, or we could search for other cyclic fraction sets when  $n > 20$ . Students might shy away from the latter approach because of the work involved in long division, but there is a way to use a calculator to shorten the work. Using a calculator, we can get the beginning of each expansion. For  $n = 23$  we get:

$$\begin{array}{lll}
 \frac{1}{23} = .04347826... & \frac{8}{23} = .34782608... & \frac{15}{23} = .65217391... \\
 \frac{2}{23} = .08695652... & \frac{9}{23} = .39130434... & \frac{16}{23} = .69565217... \\
 \frac{3}{23} = .13043478... & \frac{10}{23} = .43478260... & \frac{17}{23} = .73913043... \\
 \frac{4}{23} = .17391304... & \frac{11}{23} = .47826086... & \frac{18}{23} = .78260869... \\
 \frac{5}{23} = .21739130... & \frac{12}{23} = .52173913... & \frac{19}{23} = .82608695... \\
 \frac{6}{23} = .26086956... & \frac{13}{23} = .56521739... & \frac{20}{23} = .86956521... \\
 \frac{7}{23} = .30434782... & \frac{14}{23} = .60869565... & \frac{21}{23} = .91304347... \\
 & & \frac{22}{23} = .95652173...
 \end{array}$$

If  $n = 23$  gives a cyclic set of fractions, by looking at  $\frac{1}{23}$  we know that 8 digits of the cycle are 04347826. The pattern 826 must appear in another expansion, and we find it at the beginning of  $\frac{19}{23} = .82608695....$  Now we know that the next 5 digits of the cycle are 08695. The cycle can be built up in this manner until the digits start to repeat.

$$\frac{1}{23} = .04347826...$$

$$\frac{19}{23} = .82608695...$$

$$\frac{16}{23} = .69565217...$$

$$\frac{5}{23} = .21739130...$$

$$\frac{3}{23} = .13043478...$$

The complete cycle is then seen to be 0434782608695652173913. This cycle has 22 digits, so  $n = 23$  must generate a set of cyclic fractions.

We will also find that  $n = 29$  gives a set of cyclic fractions. However, when we reach  $n = 31$ , we find that the cycle is completed with only 15 digits. If we then select a fraction that does not belong to the cycle and do the same analysis, we will find a second 15-digit cycle. These two cycles will account for all 30 fractions in the fraction set for  $n = 31$ . Here is a possibility we had not considered before; one fraction set contains two distinct cycles.

In light of this discovery, perhaps we should go back and have another look at the non-cyclic fraction sets where  $7 < n < 20$ . When  $n = 11$ , we find five cycles of two digits each, and  $n = 13$  has two 6-digit cycles. We will also find that  $n = 37$  has twelve 3-digit cycles,  $n = 41$  has eight 5-digit cycles, and  $n = 43$  has two 21-digit cycles.

We observe that when a fraction set has more than one cycle, the cycles are all the same length and that the number of cycles times the number of digits per cycle is  $n-1$ . Some insight as to why this happens can be gained by looking closely at an example, say  $n = 13$ .



$$\begin{array}{ll}
 \frac{1}{13} = .\overline{076923} & \frac{7}{13} = .\overline{538461} \\
 \frac{2}{13} = .\overline{153846} & \frac{8}{13} = .\overline{615384} \\
 \frac{3}{13} = .\overline{230769} & \frac{9}{13} = .\overline{692307} \\
 \frac{4}{13} = .\overline{307692} & \frac{10}{13} = .\overline{769230} \\
 \frac{5}{13} = .\overline{348615} & \frac{11}{13} = .\overline{846153} \\
 \frac{6}{13} = .\overline{461538} & \frac{12}{13} = .\overline{923076}
 \end{array}$$

We see that  $\frac{1}{13}$  and  $\frac{2}{13}$  belong to separate cycles so let us look closely at the long divisions that give those expansions.

$$\begin{array}{r}
 .076923 \\
 13 \overline{) 1.000000} \\
 \underline{0} \\
 100 \\
 \underline{91} \\
 90 \\
 \underline{72} \\
 120 \\
 \underline{117} \\
 30 \\
 \underline{26} \\
 40 \\
 \underline{39} \\
 1
 \end{array}
 \qquad
 \begin{array}{r}
 .153846 \\
 13 \overline{) 2.000000} \\
 \underline{13} \\
 60 \\
 \underline{65} \\
 50 \\
 \underline{39} \\
 110 \\
 \underline{104} \\
 60 \\
 \underline{52} \\
 80 \\
 \underline{78} \\
 20
 \end{array}$$

The remainders when expanding  $\frac{1}{13}$  are 10, 9, 12, 3, 4, and 1, so any time the division gives one of those remainders, we will start producing the cycle 076923. Those remainders will occur at the first step in the expansions of  $\frac{10}{13}$ ,  $\frac{9}{13}$ ,  $\frac{12}{13}$ ,  $\frac{3}{13}$ ,  $\frac{4}{13}$ , and  $\frac{1}{13}$ , and these six fractions each begin with a different digit of the 6-digit cycle. The remainders 7, 5, 11, 6, 8, and 2 occur in the division for  $\frac{2}{13}$ , and the fractions  $\frac{7}{13}$ ,  $\frac{5}{13}$ ,  $\frac{11}{13}$ ,  $\frac{6}{13}$ ,  $\frac{8}{13}$ , and  $\frac{2}{13}$  each begin with a different digit of the second cycle, 153846. Each possible remainder occurs in one of the two divi-

sions, therefore the two cycles account for all the proper fractions with denominator 13.

We can now view cyclic fraction sets as a special case of a more general phenomenon. All fraction sets with prime denominators will have cyclic expansions of varying lengths. The length of each cycle will be the same, and the number of cycles times the length of each cycle will be  $n - 1$ . What made the case  $n = 7$  interesting originally was the fact that the expansions were fairly short and there was only one cycle, so the pattern was easy to see.

The teacher should be cautioned about following the sequence outlined in this paper too stringently. The exploration can take many directions, and students should be allowed to structure their own investigations. For example, students who are competent at computer programming might wish to make a computer search for cyclic fraction sets. They could find a large number of them and perhaps make some conjectures about their relative frequency. Do they occur less frequently or more frequently for larger  $n$ , or is their frequency proportional to the frequency of primes? It would be interesting to let two groups of students pursue this topic independently, and then compare their results and methods.

The initial discovery of cyclic fractions, the classification of fraction sets by types of expansions, and the method for constructing a long cycle with the aid of a calculator are discussed by Sue S. Wagner, "Fun With Repeating Decimals," Mathematics Teacher 72 (March 1979), p. 209-12. The proofs, plausibility arguments, and emphasis on the role of the remainders in the division process are the work of the author.

## CYCLIC FRACTIONS -- THEORETICAL CONSIDERATIONS

The previous chapter, "Cyclic Fractions -- Empirical Explorations," explored the occurrence of what were called cyclic fractions. Much data was generated, and some conclusions were drawn. Plausibility arguments based on numerical examples were given to support the conclusions. Adept students might be interested in further justification of the relationships that were uncovered. This chapter presents some of the number theory behind the conclusions that were drawn. Students should not attempt to study this theory without first having done some of the numerical explorations outlined in the previous unit. Deductive proofs to support the conclusions drawn require the use of congruences and the division algorithm. In the previous lesson, we decided that all fraction sets with prime denominators had one or more cycles in their expansions. For the rest of this article, let  $p$  denote a prime number.

Before, it was decided that  $p$  determined a cyclic fraction set if and only if the period of the expansion of  $\frac{1}{p}$  contained  $(p - 1)$  digits. A sensible place to begin a theoretical investigation is to see what can be determined about the length of the period of the decimal expansion of  $\frac{1}{p}$ . This expansion will begin to repeat the first time we get a remainder of 1, since that puts us back where we began. Suppose there are  $m$  repeating digits in the expansion; then the  $m$ th remainder is 1. Since the remainders dictate what happens in the division, it would seem only natural that we try to phrase this result in terms of congruences. Con-

gruences do not deal with a remainder that occurs several places after the decimal point; they deal with the remainder that is left after an integral quotient has been obtained. To get around this problem, multiply  $\frac{1}{p}$  by  $10^m$ . This shifts the decimal point  $m$  places to the right so the first  $m$  digits of the expansion (the repeating block) appear in front of the decimal point, and there is a remainder of 1 after the integral part of the division is completed. We can write  $\frac{10^m}{p} = q + \frac{1}{p}$ , or  $10^m = qp + 1$ , where  $q$  is an  $m$ -digit number. This last equation is easily rewritten as  $10^m \equiv 1 \pmod{p}$  where  $m$  is the length of the period of  $\frac{1}{p}$ .

If  $m$  is the length of the period, then a remainder of 1 occurs after  $m$  digits, but also after  $2m$  digits,  $3m$  digits, and so on. So if  $10^m \equiv 1 \pmod{p}$  then also  $10^{2m} \equiv 1 \pmod{p}$ ,  $10^{3m} \equiv 1 \pmod{p}$ , etc. We are interested in the smallest positive integer  $m$  such that  $10^m \equiv 1 \pmod{p}$ . In number theory this smallest integer is called the order of 10 modulo  $p$ .

Now generalize a bit and consider the expansion of  $\frac{a}{p}$  where  $1 \leq a < p$ . The digits will begin to repeat when we get a remainder of  $a$ , say after  $m'$  digits. Using the same reasoning as before, we can get that  $a \cdot 10^{m'} = q'p + a$ , or  $a \cdot 10^{m'} \equiv a \pmod{p}$ . A property of congruences is that if  $a$  and  $p$  are relatively prime, we can cancel a factor of  $a$  from each side of the congruence. Since  $1 \leq a < p$  and  $p$  is prime,  $a$  and  $p$  must be relatively prime. Hence, we can get that  $10^{m'} \equiv 1 \pmod{p}$ . Again, we get that  $m'$  is the order of 10 modulo  $p$ , so the period of the expansion of  $\frac{a}{p}$  is the same as that of  $\frac{1}{p}$ . This explains why, in a fraction set with

two or more cycles; all the cycles are the same length.

If we want to predict the length of the period in the expansion of  $\frac{1}{p}$ , we must determine the smallest integer  $m$  for which  $10^m \equiv 1 \pmod{p}$ . We could simply try successive values for  $m$  until we find one. There is, however, a way to restrict the possible values for  $m$ . An important theorem in number theory is Fermat's Little Theorem, which says that if  $p$  is prime and  $a$  and  $p$  are relatively prime, then  $a^{p-1} \equiv 1 \pmod{p}$ . Applying this theorem, we have that  $10^{p-1} \equiv 1 \pmod{p}$ . Recall the long division which is used to expand  $\frac{1}{p}$ . If the period is  $m$  digits, a remainder of 1 occurs after  $m$  digits, after  $2m$  digits, and so on. Furthermore, these are the only places that a remainder of 1 occurs, since the appearance of 1 signals the start of a new period. Therefore,  $10^y \equiv 1 \pmod{p}$  if and only if  $y$  is a multiple of  $m$ . Since  $10^{p-1} \equiv 1 \pmod{p}$ , we can conclude that  $p - 1$  is a multiple of  $m$ , or equivalently, that  $m$  is a factor of  $p - 1$ .

This property reduces the work involved in finding the period of a fraction with prime denominator. Suppose we wish to determine the period of  $\frac{1}{17}$ . We must have that  $m$  is a factor of 16, or  $m = 1, 2, 4, 8$ , or 16.  $10 \equiv 10 \pmod{17}$ ,  $10^2 \equiv 15 \pmod{17}$ ,  $10^4 \equiv 4 \pmod{17}$ , and  $10^8 \equiv 16 \pmod{17}$ , so by the process of elimination we know that  $10^{16} \equiv 1 \pmod{17}$  and the period of the expansion of  $\frac{1}{17}$  is 16 digits long. We know, then, that  $p = 17$  determines a fraction set with one cycle, or what we originally called a cyclic fraction set.

Now investigate the length of the period of  $\frac{1}{41}$ . The length of the period is a factor of 40, so  $m = 1, 2, 4, 5, 8, 10, 20$ , or 40. Trying

these possibilities in order, we get  $10 \equiv 10 \pmod{41}$ ,  $10^2 \equiv 18 \pmod{41}$ ,  $10^4 \equiv 37 \pmod{41}$ , and  $10^5 \equiv 1 \pmod{41}$ ; so the expansion of  $\frac{1}{41}$  has a 5-digit period. We also know that all proper fractions with denominator 41 have 5-digit periods. Each 5-digit cycle accounts for 5 of the fractions, so 8 separate cycles are needed to take care of all 40 fractions. To see which fractions belong to which cycle, we can study the division required to expand one of the fractions. For instance,

$$\begin{array}{r}
 .02439 \\
 41 \overline{) 1.00000} \\
 \underline{0} \\
 100 \\
 \underline{82} \\
 180 \\
 \underline{164} \\
 160 \\
 \underline{123} \\
 370 \\
 \underline{369} \\
 1
 \end{array}$$

This cycle will be entered when a remainder of 10, 18, 16, 37, or 1 occurs, so the fractions  $\frac{10}{41}$ ,  $\frac{18}{41}$ ,  $\frac{16}{41}$ ,  $\frac{37}{41}$ , and  $\frac{1}{41}$  belong to this cycle.

If the order of 10 modulo  $p$  is  $p - 1$  then 10 is called a primitive root modulo  $p$ . We can now state that  $p$  generates a cyclic fraction set (i.e. a fraction set whose expansions all belong to the same cycle), if and only if 10 is a primitive root modulo  $p$ . It can be proven that every prime has at least one primitive root, but there is no known method for predicting what the roots will be. To find them, it is necessary to use trials. It is also not known whether the set of primes having 10 as a primitive root (and hence generating cyclic fraction sets) is infinite or how these primes are distributed.

It is possible to extend some of the results derived here to more general theorems. For instance, by referring to the division process, it was possible to show that  $10^y \equiv 1 \pmod{p}$  if and only if  $y$  is a multiple of the order of 10 modulo  $p$ . A generalization of this result is a well known theorem in number theory:  $a^y \equiv 1 \pmod{p}$  if and only if  $y$  is a multiple of the order of  $a$  modulo  $p$ . This can be justified by considering a division in base  $a$ . In fact, this study can serve as a motivation to introduce a number theory unit in congruences.

It is a small step to extend the analysis of the periodicity of decimal expansions to fractions with composite denominators which do not include a factor of 2 or 5. The analysis of the length of the period of  $\frac{1}{p}$  did not ever use the fact that  $p$  was prime. Therefore, this same argument can be used to show that for any number  $n$ , relatively prime to 10, the period of the expansion of  $\frac{1}{n}$  is the order of 10 modulo  $n$ . In generalizing the argument to the proper fraction  $\frac{a}{p}$ , we did use the fact that  $p$  was prime, but only to show that  $a$  and  $p$  must be relatively prime. If we restrict ourselves to only those fractions  $\frac{a}{n}$  which cannot be reduced, then  $a$  and  $n$  are relatively prime, and we can again say that the period of  $\frac{a}{n}$  is the same as that of  $\frac{1}{n}$ .

Upon first glance, it might seem that there is nothing in the arguments to prevent applying them to  $\frac{1}{n}$  where  $n$  includes factors of 2 or 5, i.e. when  $n = 2^r 5^s n_1$ . The problem is that in this case there is no number  $m$  such that  $10^m \equiv 1 \pmod{2^r 5^s n_1}$ . If  $r \neq 0$ , this would imply that  $10^m = q(2^r 5^s n_1) + 1 = 2(q2^{r-1} 5^s n_1) + 1$ , which indicates that  $10^m$  is odd, a contradiction. If  $r = 0$  then  $s \neq 0$  so we can argue that  $10^m =$

$q(5^s n_1) + 1 = 5(q5^{s-1} n_1) + 1$ , also a contradiction. What this means in terms of the long division process is that a remainder of 1 does not occur. If  $n_1 \neq 1$  the fraction must repeat, so the repeating block must begin sometime after the first position, what we called a delayed repeater. An example can be seen in the expansion of  $\frac{1}{14}$ .

$$\begin{array}{r}
 .0714285 \\
 14 \overline{) 1.000000} \\
 \underline{0} \\
 100 \\
 \underline{98} \\
 20 \\
 \underline{14} \\
 60 \\
 \underline{56} \\
 40 \\
 \underline{28} \\
 120 \\
 \underline{112} \\
 80 \\
 \underline{70} \\
 10
 \end{array}$$

In general, it can be shown that if  $\frac{a}{n}$  is a reduced fraction and  $n = 2^r 5^s n_1$ , then the decimal expansion of  $\frac{a}{n}$  has period equal to the exponent to which 10 belongs modulo  $n_1$ , and there are  $t = \max\{r, s\}$  non-repeating digits in front of the repeating block. A proof can be found in James E. Shockley, Introduction to Number Theory (New York: Holt, Rinehart & Winston, 1967), p. 93.



## CONCLUSION

Teaching problem solving requires not only finding a solution to a problem, but more importantly, it requires verbalizing how that solution was obtained. Students should be told what assumptions were made and why they were necessary, what directions were tried that did not prove successful, how the successful direction was discovered, how previously learned knowledge was synthesized into the solution, and how the results were evaluated for reasonability and consistency. It is difficult to become accustomed to verbalizing all the thought processes that lead to a solution. The teacher must become adept at this type of explanation so the students can understand the entire process of solving a problem. In addition, the teacher should require the students to do more than simply report answers. They should be able to give oral or written explanations of how they arrived at those answers. The detail and depth of the explanations should increase as the students become more accustomed to this task and more adept at mathematics. Such activity will take a considerable amount of time; it is simply not possible to move at the rapid pace that a lecture-style of teaching permits.

The problems in this thesis are ones that could be used for a unit in problem solving or as supplementary problems integrated into the regular curriculum. The author has attempted to discuss the heuristics of the solutions and to indicate how these heuristics could be developed

in a classroom. The skills required for successful problem solving are gained slowly and require continual reinforcement and expansion. Therefore, the teacher is urged to transfer problem solving techniques to teaching the rest of the curriculum. Once a strategy has been introduced, it should be pointed out whenever it is subsequently used. In a sense, then, problem solving becomes a teaching style rather than a teaching topic.

In a problem solving style of teaching, much emphasis is put on alternate methods of solution. The teacher must be continually receptive to suggestions for alternate solutions from the students. Even approaches which will not lead to solutions should be fully examined to see why they fail. If all suggestions are accepted and considered, students will be more inclined to offer them and thus more ideas about a particular situation will be generated.

It has been the author's experience that teaching problem solving is fun, both for the students and the teacher. It requires some re-orientation of the traditional picture of students raising their hands one at a time to ask questions while others work quietly in an orderly classroom. Like any readjustment of the teaching process, a problem solving style of teaching requires some individual fine-tuning before any teacher will feel comfortable with it. The potential of having students be more actively involved in the discovery of mathematics would seem to justify that effort.

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